Electoral fraud and voter turnout

Vardan Baghdasaryan

Graduate School in Economics and Finance of Public Administration

Catholic University of Sacred Heart

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Abstract

We model a costly voting game over two candidates where voters can either participate favouring their preferred candidate or abstain. Unlike conventional models, the electoral process can be influenced by one of the candidates to insure a favourable income for him/her. Electoral fraud mechanisms are assumed to be of two types: affecting directly voting costs of individual strategic voters and those affecting pivot probabilities without dealing with voting costs. Rather than explicitly modelling a game played by the voters and the falsifying candidate, special parametrization is applied so that the level of fraudulent intervention is determined endogenously in the equilibrium of the game played among the voters and its magnitude depends on the targets of the falsifier. Assuming that the number of the voters is unknown and they have got only incomplete information about the voting costs, it is demonstrated that depending on the goals of the falsifying candidate the fraud may actually increase the voter turnout. In particular, if the fraud applied is of a magnitude to ensure for the falsifying candidate a victory at the margin, then the participation rates of the voters sustaining both alternatives are increased compared to the no fraud situation. These results hold independently of the type of the intervention applied. At the same time, the strategic motives to vote diminish at high rate once the fraud goes beyond the level guaranteeing simple victory in binary elections.

1 Introduction

Traditionally the election participation decisions made by the strategic voters have been analysed for fully democratic elections which unconditionally respect "one voter - one vote" rule and do not allow for any kind of intervention into the electoral process. At the same time, around 30% of all national level elections around the world during the 1975-2010 were characterized by fraudulent intervention[1]. It is important to investigate the possibilities of applying the strategic voting models taking into account imperfect organization of the electoral process. In particular, can the canonical models of voter turnout explain variations in the turnout due to the fraud and is it possible to get an additional insight into the effect of the fraud on the participation decisions of the voters? A simplified approach to the problem suggests that fraud during elections suppresses participation motivation as one to one mapping from the voter to the vote is no more valid and the election outcome is less dependent on the expressed will of the citizen-agents. In this research paper we demonstrate that depending on the type and scale of the fraudulent interventions and the distribution of preferences among the voters equilibrium turnout may also increase due to the fraud.

But why should we bother ourselves with the impact of fraud on the voter turnout? The answer to this question consists of several points and underpins our motivation to pursue this topic. Firstly, it is important to analyse whether there are types of electoral fraud which in spite of limited direct impact can considerably change the behaviour of strategic voters. Secondly, it is interesting to investigate the conditions that decrease or increase the actual and observed participation rates. Thirdly, the predictions of the model developed must be in accordance with the observed relationship between the fraud and the turnout in various regions of the world. Methodologically, this research aims to apply strategic voter turnout model of completely fair elections to a setting in which one participant of the electoral process has an access to the (costly) mechanisms of influencing the elections' outcome through illicit and fraudulent undertakings.

We believe that consideration of this topic builds on the long-lasting discussion in the political economy literature on the "Paradox of voting" - why do people vote if their individual impact on the outcome is negligible. Under the assumption of instrumental voting by the agents the utility of a citizen who votes has the following form:

$$U = p * B - c \tag{1}$$

where p - is the probability of casting a decisive vote in the elections, B - is the benefit associated to the preferred outcome and c - is the cost of voting, which can have various interpretations such as time spent to vote or time and resources spent to find out information about the alternatives. It is argued that with the increase in the number of voters the probability of casting a decisive vote tends to zero and nobody will eventually vote given that this action is costly - giving rise to the above mentioned paradox. One simple way to get out of the presumed deadlock is to consider the possibility of an

[1] Author's calculations based on the data coming from database on political institutions (DPI data available from mid of 1970s) and the Cingranelli-Richards human rights dataset (CIRI data available from 1981).

additional source of benefit from the act of the voting, such as utility derived from the exercise of the citizen duty. This may turn the cost of voting negative and explain mass turnout at the large elections with negligible p, hence almost no instrumental benefit (Downs, 1957).

In 1983 in their seminal paper Palfrey & Rosenthal state the “paradox” is in fact a logical fallacy which arises because the probability of casting a decisive (pivotal) vote and the probability to vote of an individual voter were considered to be independent, whereas these two are determined as an outcome of a simultaneous decision process. Recognizing this simultaneity and formalizing a game with complete information where the voters are playing in a non-cooperative manner the authors demonstrate existence of Nash equilibria with substantial turnout rates, that is in equilibrium the presumably small “p” in equation (1) could in fact be very close to 1, independent of the electorate size (Palfrey & Rosenthal, 1983). At the same time these results required quite strong assumptions, such as unique level of the voting costs for all the voters. Relaxing this assumption and introducing incomplete information over the heterogeneous voting costs, the authors came to an unsatisfactory conclusion that only voters with zero or negative voting costs will participate in elections with large number of the voters and that there cannot be strategic motives behind considerable turnouts in large elections (Palfrey & Rosenthal, 1985).

Generally overall development of the literature in this field since then can be conditionally divided in two main streams: research papers trying to solve rigorously the “Paradox of voting” and rationalize the mass turnouts at large elections and the researches which use the costly strategic voting setting to analyse various factors of the electoral process such as role of information or chances of the minority to win the elections.

Though the current research project is not an attempt to solve the paradox of voting, we still find it important to briefly review some of the methods proposed in the recent literature to justify the possibility of massive turnout when the population of potential voters is large. Intuitively one can assume that to overcome the low turnout problem the benefits of the participation must be somehow justified to be much higher than the voting costs are. Thus, Faravelli and Walsh (2011) propose to use the idea of paternalistic voters which enjoy additional benefits based on the fact that other voters will be governed by the candidate they prefer. Combining the paternalistic voters with what they call smooth policy function the authors show the possibility of considerable turnout in large elections. Smooth policy function or importance of the mandate (Castanheira, 2003) replaces the winner take-it-all assumption and the amount of the benefit depends on the share of votes obtained by the candidate (including the candidate who lost the election). Myatt (2012) argues that the introduction of aggregate uncertainty about the voters' preferences can explain mass turnout at the elections with large number of voters. Still his results require also very small cost-to-benefit ratio for the voters and the author again proposes the idea of paternalistic voters to justify the tiny cost-to-benefit ratio of the voting agents.

Modelling the voters' actions as strategic gives additional ways of analysing different aspects related to the elections. Issues like the chances of the minority to win the elections, the role of the opinion polls, compulsory voting rule impact on the outcome are among the ones addressed by the researchers applying the costly strategic voting setting for their analysis. Thus it has been demonstrated that minorities can have higher probabilities of winning in binary elections. Particularly, this result can arise due to differences in the intensity of preferences in favour of the minority voters (Campbell, 1999) or to the lower impact of free-riding problem in the minority group (Haan & Kooreman, 2003). Problem of excessive turnout is investigated by Borgers (2004), who shows that when distribution of political preferences is symmetric, equilibrium voter turnout is excessive and thus compulsory-voting laws make little sense. Another study refers to the role of pre-election polls and the possibility of the strategic behaviour by the respondents given they act strategically also when deciding to participate in the costly voting (Taylor & Yildirim, look in public choice journal for the date). Their results show that strategic voting brings to strategic behaviour in responding the pre-election surveys and the misleading reporting of preferences is actually optimal behaviour for the large electorates.

Our research agenda is closer to the second group of literature presented above. We believe that modelling voting decisions as strategic and analysing the fraudulent intervention into the electoral process in such an environment can both explain facts observed about turnout in developing and transition countries, as well as highlight the channels through which the fraud affects the voters' decisions.

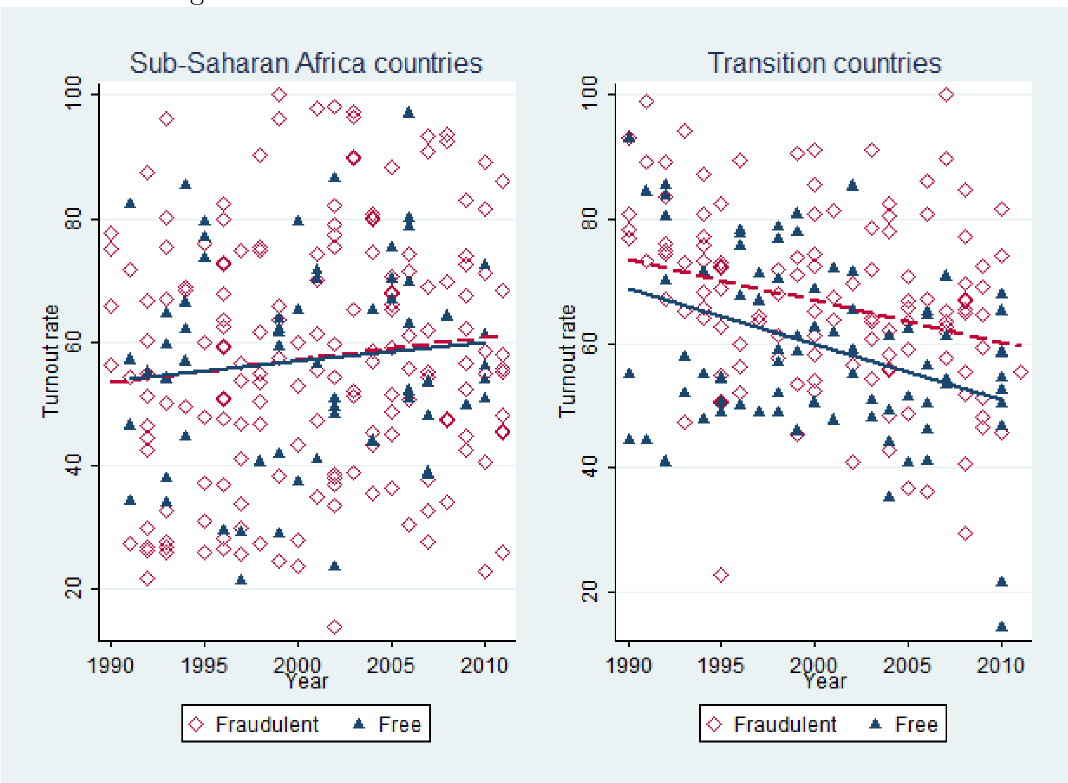
Fraud during elections, as already mentioned at the beginning, is a widely applied tool by the semi-democratic or transition regimes to secure their power and at the same time imitate the process of national bodies' democratic formation. For our purposes it is important to have at least some classification of the illicit interventions as our ultimate goal is to model these actions in a specific setting. Thus we can classify the electoral fraud and/or illicit actions into the following three broad groups: a) limitation of the ability of a voter to express his preferences by exercising his right to vote, b) manipulation of the results during or after the elections without any direct interference with the voters, and c) cooperation with voters in organizing and conducting illicit practices during the election process (see Jones 2005 for more detailed treatment of the problem). The first group may include such actions as limitation of access to voting stations by intimidation, violence or other selective challenges to "undesirable" voters. Those voting forced to do so are also considered to have limited decision making possibilities and can be attributed to the first group. The second group may include ballot box stuffing before, during or after the completion of elections, substitution of counterfeit ballot box for authentic box or ballot alteration. Finally, examples of actions from the third group may be one person registering in many places, voting using illegal registration, chain voting and, last but not least - vote buying. In spite of its often occurrence, fraud hasn't been widely analysed in the theoretical literature on participation to voting. But still a number of related research questions has been raised. Thus vote buying has been analysed as a legal phenomenon being adopted by the both competitors in the elections and it is shown that parties spent less with buying the votes compared to the situation when they would have to compete on campaign promises (Dekel et al. 2008). But the question of voter turnout is not analysed in this paper. Possibility of post-electoral violence is analysed by Ellman and Wantchekon (2000) and the authors demonstrate that in a situation without policy commitment and asymmetry of information about the probability of unrest the voters select the strong party to avoid disruptions. In a more recent paper Collier and Vincent (2011) analyse the difference of electoral fraud and electoral violence in the political economy context of Sub-Saharan Africa, demonstrating that in case the political competitors are weak they are most likely to resort to violence, whereas when they are strong they prefer to use bribing and ballot fraud. The game is modelled between the incumbent and the challenger, whereas we will try to tackle very similar problem with strategic voters.

Electoral fraud has been analysed also in empirical literature. A recent study investigates causal relationship between the fraud during the election and the probability of the incumbent to win such elections - efficiency of fraud. According to the results obtained analysing 786 elections in 155 country, fraud during elections increases the probability of the incumbent to win from 62% up to 84% (on average) (Collier & Hoeffler, 2009). But this study doesn't address the interconnection of fraud and voter turnout at the elections. One of the few papers using fraud as a determinant of voter turnout finds it to be a significant negative predictor of the turnout (Vergne, 2009). This analysis covers 60 developing countries during the period of 1980-2005 and the the microeconomic foundation of fraud is that it increases the cost of participation by making the outcome less predictable. Vote buying has received considerably more attention in empirical literature (see Schaffer 2007 for a survey of vote buying in different developing countries). Relation of vote buying and voter turnout in a research carried out for Nigeria based on voters' representative survey revealed no statistically significant impact of vote buying, whereas threat of violence has a negative effect on turnout (Bratton, 2008). However, it is not fully clear whether those who were approached with the offer of bribe eventually accepted it and whether the intimidation was exercised to make them vote or abstain.

In a related empirical paper we try to analyse the impact of electoral fraud on the observed turnout rates at national level elections in the world. Though there is no information about the types of illicit intervention in different countries, the results obtained suggest the possibility of both adverse and positive impacts on turnout rates (Baghdasaryan, 2012). Figure 1 provides a simple first approach relationship between fraud and the voter turnout rates for the Sub-Saharan Africa (SSA) and Transition

Economies. While in Sub-Saharan Africa there is no difference in the turnout rates between the fair and fraudulent elections, for the Transition countries we do observe a clear pattern - fraud is associated with higher turnout rates. One of our goals is to try to explain these patterns, keeping in mind that fraud in SSA countries is more often associated with instances of violence and intimidation, whereas in transition countries more sophisticated techniques are applied.

Figure 1: Turnout rate: democratic vs fraudulent elections



In what follows, we model a costly voting game over two candidates where voters can either participate favouring their preferred candidate or abstain. Rather than explicitly introducing the third player - the falsifying Candidate - we use additional parameters in the model to let the fraudulent intervention arise in the equilibrium of the model. Still we present the possible pay-off functions of the falsifying candidate that may rationalize the intervention, assuming that fraud is possible only as a result of costly effort and it should result in the shift of the expected plurality in the falsifier's favour.

First we set out the model. Then we analyse the equilibrium assuming complete information and incomplete information accordingly. Whenever analytical results are not enough to come up with a conclusion we present the outcomes of the numerical simulations. We conclude the discussion by summarizing the results obtained and suggesting directions for further research.

2 The Model

Consider a game in which voters must select between 2 alternatives (candidates) and one of the candidates has the possibility to interfere in the game. There are N agents who can choose to vote or abstain. A share γ of these agents prefer the Incumbent I and the remaining $1-\gamma$ prefer the Challenger C. Whenever N is known to the voters let $N_C = (1-\gamma)N$ and $N_I = \gamma N$. If not stated otherwise, we will assume that $\gamma < 0.5$, that is the supporters of the Incumbent are the minority group.

The strategy of a representative voter i supporting the Challenger is a probability distribution $\sigma_i^C \ \epsilon \ \Sigma_i^C$. Given that the set of actions of the voter i is either participation ($s_i = 1$) or non-participation ($s_i = 0$), it directly follows that $\sigma_i = 0$ corresponds to abstention (non-participation), $\sigma_i = 1$ refers to participation and the mixed strategy of the voter is just his probability of voting. The same applies for voter j supporting the Incumbent I, so her strategy is again a probability distribution $\sigma_j^I \ \epsilon \ \Sigma_j^I$. To economize on notations let $\sigma_i^C = \sigma_{Ci}$ and $\sigma_j^I = \sigma_{Ij}$.

In our analysis we will focus on team symmetric mixed strategy equilibrium (TSE).

Definition: Team symmetric equilibrium (TSE) in mixed strategies is an equilibrium in which all the voters supporting the same candidate apply the same strategy $\Rightarrow$ $\sigma_{Ci} = \sigma_C \ \forall \ i \ \epsilon \ N_C$ and $\sigma_{Ij} = \sigma_I \ \forall \ j \ \epsilon \ N_I$ and $\sigma_C \ \epsilon \ (0,1)$, $\sigma_I \ \epsilon \ (0,1)$.

We assume that the incumbent can take actions, illicit and/or fraudulent by nature, which should shift the expected plurality of the voting game in his favour. Moreover, we assume that the challenger doesn't have the resources to undertake counteractions which can have any material impact on the outcome of the elections[2]. Different types of fraud and/or illicit actions described in the previous section for our current purposes and according to the logic of the model being developed here can be classified into following two groups:

1. Direct change in the number of voters voting with certainty. One can think about this as of a situation in which a subset of voters supporting the incumbent are forced or bribed to vote at the elections day. An alternative approach would be to assume that ballot box stuffing with fake ballots is applied by the forces supporting the incumbent.

2. Increase/decrease in the cost of voting for C/I respectively.

[2]In many countries with weak rule of law and the low level of separation of powers the ruling party (elite) exercises its power to obtain an advantage during the electoral process. The opposition (challenger) usually is restricted in applying similar techniques by the fear of punishment which will be selectively applied only to one party of the electoral process - the challenger. That's the reason we find it reasonable and grounded to give only to the incumbent the possibility of intervention and to leave no role for the challenger

Based on this classification we define the strategy of incumbent politician as a pair (l, β) - $[0, L] \times [1, 0]$. 'l' is the number of voters supporting the Incumbent voting with certainty (which can be both out of the existing pool of voters as well as additional voters (ballots) and L is the maximum number of certain voters or counterfeit ballots available). 'β' is the coefficient by which the voting costs of the Incumbent's supporters are decreased and the way this decrease works obviously will differ in the cases of unique (common for all the voters) and heterogeneous voting costs[3].

Players' preferences: The utility of a representative voter j supporting the Incumbent is formulated as follows:

$$U^j = p(l, \bullet)B - c_j(\beta) \tag{2}$$

where

p - is the probability of casting a pivotal vote, and $p(l, \bullet)$ indicates that apart from other variables and parameters this probability is also a function of the 'certain' voters brought in by the Incumbent.

B - is the payoff which the voter gets when her preferred outcome is the winner (for simplicity we normalize it to 1),

c_j - is the voting cost of voter j. Given the normalization of B, term c could be considered as the cost-to-benefit ratio of the voter. Note that by definition of β $c_j(\beta)$ is decreasing in β.

The utility of a representative voter i supporting the Challenger is exactly the same apart from the voting cost part which is not dependent on the β as it was formulated above.

$$U^i = p(l, \bullet)B - c_i \tag{3}$$

Though we do not explicitly use the utility of the Incumbent to derive the equilibrium of the model, still we find it important to present the way his preferences could explain the need to interfere into the electoral process. The utility of the Incumbent can be formulated in the following two ways:

1. If the Incumbent is interested in the marginal increase of the votes in his favour:

$$U^I = R_1 * EP(l, \beta) - C(l, \beta) \tag{4}$$

[3]Note that an alternative strategy specification is possible when the voting costs of the opponent group are increased. We will denote this strategy by $\alpha \ \epsilon \ [1, \bar{C}]$

2. If the Incumbent is interested in getting just more votes than the opponent:

$$U^I = -C(l,\beta) + \begin{cases} R_2 & if \quad EP(l,\beta) > 0 \\ 0 & if \quad EP(l,\beta) \leq 0 \end{cases} \tag{5}$$

where

EP - is the Expected Plurality and is defined as the difference between the expected number of votes cast in favour of the incumbent and the expected number of votes cast in favour of the challenger $EP = l + \sigma_I[\gamma N - l] - \sigma_C(1-\gamma)N$ [4];

R_1/R_2 - is the Rent the politician gets from the marginal increase in the expected plurality/winning the election due to the illicit intervention undertaken;

C(l,β) - cost function associated with the fraudulent and illicit actions undertaken by the incumbent to increase the EP in his favour. We assume that the cost function is convex with respect to both of the intervention variables, suggesting that a deeper intervention comes at an higher cost.

The question which of the two utility functions better represent the preferences of the Incumbent doesn't have a conclusive answer. Though the theory suggests that in a binary election the Incumbent's interest must be limited by the mere fact of having more votes than the opponent, there exist wide evidence that the Incumbents apply larger scale manipulations to win the elections with higher margin. Benefits of excessive fraud during the elections can be of various types, such as discourage opponents from joining or supporting rival parties, from voting, or from participating in other ways, or on the opposite, motivate supporters to participate more actively (Simpser, 2008).

The players move simultaneously in the game considered. While for the voters in the turnout models simultaneity is considered a natural modelling approach, actions of the incumbent require additional clarification. An alternative view would be to assume that the incumbent can actually monitor the process of election (e.g. by applying exit polls) or even falsify the results after the vote counting. Instead here we assume simultaneity and ground such an approach on the fact that interventions on behalf of the incumbent need to be planned and designed beforehand and can hardly be considered as a "last minute" response to negative developments during the election day.

It is immediate to see that the decision of the Incumbent to interfere into the electoral process and to what extent to interfere depends on the assumptions about the benefit and cost parameters associated with the two different types of intervention. To derive the equilibrium of the voting game in the following sections we will focus only on the strategies of the voters. Whenever applicable, we will introduce an additional parameter,

[4] Obviously this formulation of expected plurality is for the case when the 'certain' voters for the Incumbent are coming out of the voters' pool supporting him.

$\mathcal{M}$, which will represent the required difference in votes to be achieved as a result of intervention and will make an assumption that there exist cost and benefit parameters for the Incumbent's utility function that support the level of $\mathcal{M}$ considered. Given this parametrization, the fraud level variable will be determined endogenously in the equilibrium of the model. The described approach works well whenever we work with the incomplete information version of the model. As it will be made clear below, the complete information model doesn't allow to apply the same strategy and we will have to consider the fraud as a parameter of the model, trying to carry out a comparative statics exercise and to check ex-post the rationale for the Incumbent to commit the fraud. With respect to the last point, what we need is just to assure ourselves that the intervention type applied is really beneficial for the Incumbent. This approach simplifies the subsequent analysis. Our primary goal is to see what is the total turnout and within different subgroups participation responses towards the different type of illicit actions implemented by the Incumbent.

2.1 Complete information case

We start with an analysis of a game in which all the players possess complete information so the equilibrium concept applied to solve this problem is Nash equilibrium.

We focus on team symmetric mixed strategy equilibria, as an equilibrium in pure strategies, as shown in (Palfrey, Rosenthal (PR) 1983), exists in very limited cases which require additional conditions on the relative sizes of the two groups.

Before proceeding we present the notations used below:

n_I - number of voters supporting the Incumbent who vote (out of total $\gamma * N$)

n_I^i - number of voters supporting the Incumbent who vote, but excluding voter i

n_C - number of voters supporting the Challenger who vote (out of total N)

n_C^j - number of voters supporting the Challenger who vote, but excluding voter j

To formalize the necessary and sufficient conditions for the characterization of the mixed strategy equilibrium of the game it is required to decide what is the rule applied to break the tied outcome of the elections. Following assumption is made:

Assumption 1: *Ties of the voting game are broken following the "status-quo rule"*[5]*:*

$$n_C > n_I \Rightarrow \textit{candidate } C \textit{ wins the elections}$$

[5]It is important to underline that the "status-quo rule" applied here results in simpler equilibrium conditions compared with more met "coin toss" tie breaking rule, and all the results obtained below would go through with the alternative assumption as well.

$$n_C \leq n_I \Rightarrow \textit{candidate } I \textit{ wins the elections}$$

Now we take a closer look at equation (2) keeping in mind the Assumption 1. Thus, for the voter i supporting the Incumbent following holds:

$$\text{if she votes} \Rightarrow U_V^i = 1 * prob[n_I^i + 1 \geq n_C] + 0 * prob[n_I^i + 1 < n_C] - c_i$$
$$\text{if she abstains} \Rightarrow U_A^i = 1 * prob[n_I^i \geq n_C] + 0 * prob[n_I^i < n_C]$$

Clearly the voter is indifferent between voting and abstaining when $U_V^i = U_A^i$, therefore he plays a mixed strategy $\sigma_{Ci} \, \epsilon \, (0,1)$. So in order σ_{Ci} to be the optimal strategy of a voter i supporting the Incumbent the following statement must be true:

$$prob[n_I^i + 1 = n_C] = c_i \tag{6}$$

Similarly for voter j supporting the Challenger we have that:

$$\text{if she votes} \Rightarrow U_V^j = 1 * prob[n_C^j + 1 > n_I] + 0 * prob[n_C^j + 1 \leq n_I] - c_j$$
$$\text{if she abstains} \Rightarrow U_A^j = 1 * prob[n_C^j > n_I] + 0 * prob[n_C^j \leq n_I]$$

so to have $\sigma_{Ij} \, \epsilon \, (0,1)$ as best response following must hold:

$$prob[n_C^j = n_I] = c_j \tag{7}$$

Left hand sides of equations (6) and (7) are binomial probabilities. To characterize them we start with a simple example. At the same time here we consider the type of fraud with the l voters out of total N_I supporting the incumbent vote with certainty (alternatively, voting non-strategically).

Example. Let $N_C = 4, N_I = 3$ and $l = 2$. So in this example two out three voters supporting the Incumbent are forced to vote with certainty. Alternative explanation would be that there is one 'existing' supporter of the Incumbent and two ballots are expected to be put into the box illicitly.

We start with the case when the voter j supporting the Challenger by participation breaks a tie. That is:

$prob[n_C^j = n_I] = prob[\text{“}n_C^j = 2\text{”} = \text{“}n_I = 2\text{”}] + prob[\text{“}n_C^j = 3\text{”} = \text{“}n_I = 3\text{”}]$

In following two lines focus on possible combinations and not the probabilities.

Number of combinations $[\text{“}n_C^j = 2\text{”} = \text{“}n_I = 2\text{”}] = \binom{\text{“}N_C - 1 = 3\text{”}}{\text{“}k + l = 2\text{”}} \binom{\text{“}N_I - l = 1\text{”}}{\text{“}k = 0\text{”}}$

Number of combinations $[\text{“}n_C^j = 3\text{”} = \text{“}n_I = 3\text{”}] = \binom{\text{“}N_C - 1 = 3\text{”}}{\text{“}k + l = 3\text{”}}\binom{\text{“}N_I - l = 1\text{”}}{\text{“}k = 1\text{”}}$

Thus, taking into account also voting probabilities we obtain:

$$prob[n_C^j = n_I] = \sum_{k=0}^{N_I - l} \binom{N_C - 1}{k + l}\binom{N_I - l}{k}\sigma_C^{k+l}(1-\sigma_C)^{N_C-1-k-l}\sigma_I^k(1-\sigma_I)^{N_I-l-k}$$

Analogously, probability that voter i from the group supporting the Incumbent creates a tie can be formulated as follows:

$prob[n_I^i + 1 = n_C] = prob[\text{“}n_I^i = 2\text{”} = \text{“}n_C = 3\text{”}]$

Number of combinations $[\text{“}n_I^i = 2\text{”} = \text{“}n_C = 3\text{”}] = \binom{\text{“}N_C = 4\text{”}}{\text{“}k + l + 1 = 3\text{”}}\binom{\text{“}N_I - l = 1\text{”}}{\text{“}k = 0\text{”}}$

Thus:

$$prob[n_C^i + 1 = n_I] = \sum_{k=0}^{N_I - l} \binom{N_C}{k + l + 1}\binom{N_I - l}{k}\sigma_C^{k+l-1}(1-\sigma_C)^{N_C-1-k-l-1}\sigma_I^k(1-\sigma_I)^{N_I-l-k}$$

Before proceeding with the analysis we make an assumption about the voting costs:

Assumption 2: *$c_i = c_j = c$ for any i, j.*

Unlike the assumption about the tie-breaking rule, the assumption about the unique level of voting costs for all the agents is quite restrictive. In the next section of the paper we will relax it and consider a model with heterogeneous voting costs. As it will be made clear below Assumption 2 is necessary for obtaining the equilibrium condition of the complete information model.

Now, having adopted Assumption 1 and Assumption 2 the necessary and sufficient conditions for (σ_1, σ_2) to be TSE (team symmetric equilibrium) equilibrium in this status-quo rule voting game, are:

$$prob[n_I^i + 1 = n_C] = c \tag{8}$$

$$prob[n_C^j = n_I] = c \tag{9}$$

Moreover, taking into account the binomial probabilities obtained through the generalization of the example constructed above, the equilibrium conditions (8) and (9) could be restated as follows:

$$c = \sum_{k=0}^{min[N_C-1, N_I-l]} \binom{N_C - 1}{k + l}\binom{N_I - l}{k}\sigma_C^{k+l}(1-\sigma_C)^{N_C-1-k-l}\sigma_I^k(1-\sigma_I)^{N_I-l-k} \tag{10}$$

$$c = \sum_{k=0}^{min[N_C, N_I - l - 1]} \binom{N_C}{k+l+1} \binom{N_I - l - 1}{k} \sigma_C^{k+l+1} (1-\sigma_C)^{N_C-k-l-1} \sigma_I^k (1-\sigma_I)^{N_I-l-1-k} \tag{11}$$

The results obtained so far are enough to characterize the equilibrium of the voting game considered and it is done in the Proposition 1 as follows.

Proposition 1.

1. In a costly voting game satisfying the assumptions 1 and 2 there exist a team symmetric equilibrium (TSE) of a type $(\sigma_C, \sigma_I = 1 - \sigma_C)$ and it is characterized by the following necessary and sufficient condition:

$$c = \binom{N_C + N_I - l - 1}{N_I} \sigma_C^{N_I} (1 - \sigma_C)^{N_C - l - 1} \tag{12}$$

2. $c(\sigma_C)$ is single-picked. In particular:

$$\frac{\partial c}{\partial \sigma_C} = \begin{cases} > 0 & if \quad \sigma_C < \sigma_C^* = \frac{N_I}{N_C + N_I - l - 1} \\ = 0 & if \quad \sigma_C = \sigma_C^* = \frac{N_I}{N_C + N_I - l - 1} \\ < 0 & if \quad \sigma_C > \sigma_C^* = \frac{N_I}{N_C + N_I - l - 1} \end{cases} \tag{13}$$

3. There is no Team symmetric equilibrium (TSE) of a type $(\sigma_C, \sigma_I = 1 - \sigma_C)$ which can be supported by different voting costs in the two groups of the voters.

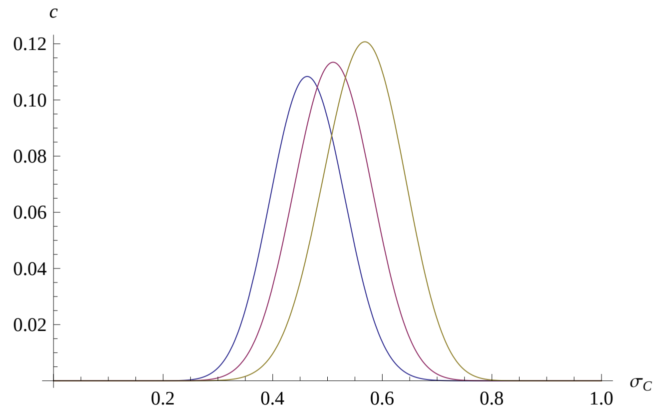
4. $\dfrac{dc_{max}(\sigma_C^*)}{dl} > 0$ for $l \leq N_I$.

5. $\dfrac{d\sigma_C}{dl} > 0$ for any fixed voting cost level c for which there exists an equilibrium with $l = 0$ and for $l << N_I << N_C$.

Proof. See Appendix.

The Figure 2 presents the findings of the Proposition 1. Still the core question of our analysis remains unanswered. Remember that l - the number of 'sure' voters brought in by the incumbent must increase his probability to win the elections. The best way to check the "usefulness" of the intervention for the incumbent would be to compare the equilibrium of the model without fraud ($l = 0$) with the equilibrium of the model with fraud ($l > 0$) at a given level of the voting cost which determines the equilibria outcomes of both models. As far as the voters' voting costs are not changed by the intervention of the incumbent (so the voting cost is the parameter of the model) this approach is quite

Figure 2: Cost c as a function of voting probability σ_C with $N_C = 30, N_I = 25$ and various level of fraud: from left to right $l = 0, l = 5, l = 10$



reasonable.

It is immediate to derive the following result which we summarize in a Lemma.

Lemma 1. Fraudulent intervention which results in subset "l" of voters from supporting the incumbent voting with certainty is beneficial for the incumbent politician in the voting game of type $(\sigma_C, \sigma_I = 1 - \sigma_C)$ if one of the following two conditions hold:

1. If the Incumbent cares only in relative increase in the number of votes in his favour:

$$l > (N_C + N_I)\frac{\sigma_C(l) - \sigma_C(l-1)}{\sigma_C(l)} \tag{14}$$

2. If the Incumbent cares only about having more votes than the opponent (in expectation):

$$l > (N_C + N_I) - \frac{N_I}{\sigma_C(l)} \tag{15}$$

given that condition (16) doesn't hold for $l = 0$, in which case for the Incumbent caring only for winning the elections there is no rationale to intervene into the elections.

■

There are two types of difficulties associated with the application of Lemma 2. First, Conditions (14) and (15) depend on the equilibrium strategies of the voters - $\sigma_C(l)$. But we know from Proposition 1 that apart from one level of voting costs in all other cases we have two equilibria for each value of the voting cost parameter. And there is no grounded approach for discriminating between the two equilibrium outcomes of the model. Second difficulty is of technical nature. The necessary and sufficient condition (12) is non-linear and obtaining the σ_C as a function of M, N, l and c in the closed form is not possible. This in its turn makes it infeasible to derive analytically the level of intervention l which would satisfy the conditions set out in Lemma 2.

However, to obtain some additional insight we apply numerical estimation techniques and the results observed are summarized in the following conjecture.

Conjecture 1.

- In case the preferences of the Incumbent are characterized by the utility function as in (3) and condition (14) is applied to assess the usefulness of the fraudulent intervention of the specific type, numerical calculations show that there exists a range of voting costs (parameter c) and a range of intervention level l such that the intervention is beneficial for the Incumbent irrespective of the fact which of the two possible equilibria outcome is applied. At the same time this is true only if the sizes of the groups are of the same size or almost of the same size and the number of the "certain" voters is close to the overall number of the Incumbent's supporters. To put it differently, almost no strategic voting is observed in favour of the Incumbent.

- In case the preferences of the Incumbent are characterized by the utility function as in (4), the negative outcome for the Incumbent can be reversed as a result of the fraud applied only for one out of two possible equilibrium outcome of the model.

- As long as the voters supporting the challenger are in majority, the introduction of "certain" voters increases the turnout, and this is without considering the very "certain" voters in the turnout rates. The probability to vote of a given voter supporting the Challenger is increasing in the volume of the fraud applied and given that the number of the voters supporting the Challenger is higher, higher is the positive impact on the turnout rates in general, in spite of the decrease in the voting probability of the voters from the minority group.

The equilibrium outcome of the complete information voting game obtained above though contains interesting results and suggests interesting behavioural response possibilities by the strategic voters, still it has got a feature hardly compatible with large elections. These equilibria are very sensitive to the number of voters involved and they are not sustained with positive voting costs even with moderate growth in the voters' number. So we continue our analysis of the fraud effect on the turnout rates by considering a modified, incomplete information version of the model with the purpose of obtaining results having more credible features.

2.2 Incomplete information case

Up to now we have been analysing the situation with homogeneous agents in terms of the voting costs. In this section we introduce two changes compared with the baseline model analysed above. First, we assume minimum level of incomplete information about the voting costs of the agents. In particular, each voter knows his own voting cost but knows only the probability distribution of the voting costs among all the agents. Thus the voting costs of the agents supporting the Incumbent (the Challenger) are independently and identically distributed according to a probability distribution function $F_I(c)$ ($F_C(c)$). By definition we have that probability of a given voter to have cost less then or equal to c is $F(c)$. For simplicity of calculations but without loss of generality we make the following assumption.

Assumption 3. The voting costs of the agents are uniformly distributed between 0 and $\mathcal{C}$:

$$c_i \sim \mathcal{U}[0, \mathcal{C}] \tag{16}$$

where $\mathcal{C}$ is the highest possible voting cost and is a parameter of the model.

The size of the population, N, in the current setting is assumed to be unknown but is commonly accepted to follow a Poisson distribution of argument λ: $\tilde{N} \sim \mathcal{P}(\lambda)$, where λ is the expected size of the population (Myerson 2000, Castaneihra 2003). As before, conditional on being picked up according to the probability distribution specified, preferences of the agents with respect to the candidates follow the same distribution as for complete information model analysed before: γ share prefer the Incumbent I and $(1-\gamma)$ share - the Challenger C. We continue with the assumption that the supporter of the Incumbent are the minority group ($\gamma < 0.5$).

Here we once again begin the analysis by considering the impact of existence of non-strategic "always voters" supporting the Incumbent. It is important to stress that unlike the case when the number of voters was known the introduction of non-strategic voters here doesn't decrease the pool of possible voters in favour of the Incumbent. In this case the ballot box stuffing with fake ballots (or same voters voting twice) may be a more intuitive way to understand the type of intervention applied.

As we know the formulation of the players' strategies in the game of incomplete information differs from the complete information game strategies. In particular it is assumed that each player has got a decision rule that he follows in the equilibrium and this decision rule maps from his own realization of voting cost to the action of voting or abstaining. As before we focus on team symmetric equilibrium justifying this approach by the fact that all the voters face the same coordination game problem and it is not too strong to assume that they would apply the same decision rule.

If we ignore Assumption 1 about the equality of voting costs among the voters, then from equations (8) and (9) it is immediate to infer that the agents i and j supporting the Incumbent and the Challenger accordingly vote if:

$$prob[n_I^i + 1 = n_C] > c_i \tag{17}$$

$$prob[n_C^j = n_I] > c_j \tag{18}$$

and abstain otherwise.

Intervention that introduces "certain" voters or ballot box stuffing

We continue the analysis assuming the first type of fraud - introduction of "certain" voters in favour of the Incumbent or, alternatively, ballot-box stuffing with counterfeit ballots. In the next subsection we will look at the other, so far not considered fraud type - introduction of asymmetry between the voting costs distribution of the voters' two groups.

Now we are ready to forumulate the Bayesian Nash equilibrium strategy of the game, which will be a pair of expected number of voters (λ_C, λ_I) which solves the following system of two equations:

$$\lambda_C = (1-\gamma) * \lambda * Prob(c_i \leq prob[\tilde{n}_C = \tilde{n}_I + l]) \tag{19}$$

$$\lambda_I = \gamma * \lambda * Prob(c_j \leq prob[\tilde{n}_C = \tilde{n}_I + l + 1]) \tag{20}$$

Using these equilibrium conditions we can state the equations characterizing the equilibrium turnout rates within each group of the voters, as well as apply a parametrization that will allow the fraud level to be endogenously determined in the equilibrium.

Proposition 2.

Let $s_C = \dfrac{\lambda_C}{\lambda}$, $s_I = \dfrac{\lambda_I}{\lambda}$, then:

$$s_C = \frac{(1-\gamma)}{\mathcal{C}} e^{-\lambda(s_C+s_I)} [\frac{s_C}{s_I}]^{\frac{1}{2}l} I_l(2\lambda\sqrt{s_C s_I}) \tag{21}$$

$$s_I = \frac{\gamma}{\mathcal{C}} e^{-\lambda(s_C+s_I)} [\frac{s_C}{s_I}]^{\frac{1}{2}(l+1)} I_{l+1}(2\lambda\sqrt{s_C s_I}) \tag{22}$$

$$l = (s_C - s_I) * \lambda + \mathcal{M} \tag{23}$$

And:

- $\mathcal{M} = 1$ if the Incumbent is just interested in winning the elections

- $\underline{\mathcal{M}} < \mathcal{M} < \bar{\mathcal{M}}$ if the Incumbent benefits from the marginal increase in EP

Proof. See Appendix.

First of all, recap that the parameter $\mathcal{M}$ here captures the margin by which the Incumbent would like to win in the elections by deploying fraudulent intervention considered. Once again, we implicitly assume that there exist cost and benefit parameters for the Incumbent's utility function that support the level intervention observed.

Proposition 2 is the transformed version of the equilibrium conditions (19) and (20). In particular, unlike the model with known number of voter where the pivot probabilities were given by binomial probabilities, these are replaced with Poisson probabilities. After some mathematical manipulations conditions (22) and (23) are obtained where $I_l(2\lambda\sqrt{s_C s_I})$ and $I_{l+1}(2\lambda\sqrt{s_C s_I})$ are modified Bessel functions (for details see Appendix). Condition (24) ensures that the fraud level is determined in the equilibrium of the model depending on the goals of the falsifier.

Remark 1. In any equilibrium characterized with positive turnout rates and $l > 0$, more voters will vote for the majority group than for the minority. These turnout rates are about those those voters who are free to choose their actions.

It is easy to verify the statement of Remark 1. From (21) and (22) we obtain that:

$$\frac{s_C}{s_I} = \frac{1-\gamma}{\gamma}\sqrt{\frac{s_I}{s_C}}\frac{I_l(2\lambda\sqrt{s_C s_I})}{I_{l+1}(2\lambda\sqrt{s_C s_I})} \tag{24}$$

The modified Bessel function is decreasing in its order and increasing in its argument (in our case - (l) and $(2\lambda\sqrt{s_C s_I})$ respectively) (see Olver 1974, p. 251), so we obtain that if $\gamma < 0.5$ (supporters of the Challenger are the majority):

$$\frac{s_C}{s_I} = (\frac{1-\gamma}{\gamma})^{2/3}(\frac{I_l(2\lambda\sqrt{s_C s_I})}{I_{l+1}(2\lambda\sqrt{s_C s_I}})^{2/3} \geq 1 \tag{25}$$

As with the complete information case, to have deeper analysis and quantify the results obtained so far we need to use numerical simulations. We start with the easy part of the problem, assuming that the Incumbent cares only about winning the election. In the following table we compare the turnout rates of the voting game without any intervention ($l = 0$) with the case of $l > 0$ and $\mathcal{M} = 1$.

The results of the numerical estimations are presented in the Table 1. The most immediate observation is that the turnout with the intervention of the Incumbent is considerably higher. At the same time it is interesting to note that when the sizes of the groups do not considerably differ from each other, then a relatively small intervention does its job quite well. Another important outcome is that the participation rates

Table 1: Turnout rate: numerical simulations

λ	10,000	10,000	10,000	100,000	100,000	100,000
γ	0.48	0.46	0.44	0.48	0.46	0.44
s_C (no fraud)	14.39%	7.48%	4.66%	3.1%	1.29%	-
s_I(no fraud)	13.64%	6.72%	3.97%	2.9%	1.16%	-
Turnout rate	28.03%	14.20%	8.63%	6.0%	2.35%	-
s_C	28.18%	29.26%	30.35%	13.08%	13.58%	14.09%
s_I	26.01%	24.92%	23.84%	12.07%	11.57%	11.07%
Turnout rate	54.2%	54.2%	54.2%	25.2%	25.2%	25.2%
Fraud level - $\mathcal{L}$	218	435	652	1008	2014	3020
Fraud rate	2.2%	4.4%	6.5%	1.0%	2.0%	3.0%
"Ex-post" fraud rate	0.8%	0.8%	0.7%	0.2%	0.13%	-

are increased both on the majority and the minority part. A possible explanation or intuition behind such a response by the voters is that the fraud of the type considered here increases the probability of tied elections. In its turn, the higher is the probability of the tied election, lower is the incentive to free ride in the larger group and the higher is the incentive to vote for the minority as the chances to win have increased for them. Note the difference in the voters' behaviour of the majority group with respect to the complete information version outcome. In the later, the "certain" voters were in fact replacing the strategic ones, which were decreasing their participation rates, whereas here we observe the opposite action.

Another important question to ask is whether the Incumbent is interested in persuading the voters that the elections will be conducted in proper way without any intervention. A possible answer to this question would be that the less the voters are confident in the democratic nature of the elections, the lower their involvement in the process will be and the easier will be to obtain the required outcome. But the results obtained through numerical estimations of the equilibrium suggest the opposite to be true - less intervention would be necessary by the Incumbent to win the elections if the voters do not internalize the possibility of the fraud. Note that the participation rates in both groups increase more or less by the same factor, but due to the numerosity of the Challenger's groups, the absolute number of voters is increasing faster than in the minority (Incumbent's) group. In other words, the response of the majority against whom the fraud is undertaken is sharper than the one of the minority, which results in the necessity of higher fraud.

The next step would be to analyse the fraud effect on the turnout rates whenever the Incumbent is just interested to have a positive shift in the Expected Plurality (EP). Note that up to now we have defined the EP for the case with commonly known number of voters. For the current formulation of the problem it would be as follows: $EP = l + s_I\lambda - s_C\lambda$. From the previous simulations we know that whenever the Challenger has got higher number of supporters and the Incumbent's intervention is just enough to win at the margin, the turnout is considerably higher than it would be without any intervention. Our goal is to see the voters' behaviour whenever the intervention is lower or higher than the one ensuring the victory at the margin. For this purpose we run the simulations fixing all the parameters of the model apart from the number of certain voters (or the number of fake ballots)[6]. As the figures 3 and 4 make it clear the highest turnout rate is achieved whenever the expected plurality is equal to zero, that is at the moment that the election outcome is a tie. When the Incumbent tries to win the election with a higher margin and pushes the number of the certain voters beyond the level just necessary to have the majority, turnout declines faster and return to its initial (no fraud) state.

The intuition behind this result is quite simple. By increase in its supporters' number the Incumbent makes the tied outcome more probable as in essence this makes the sizes of the two groups more equal. As already mentioned above, the free-riding in the larger, Challengers' group decreases and the higher chances to win for the Incumbent supporters pushes them to vote. This holds up to the moment when the expected sizes of the two voting groups equalize. Increasing the number of certain voters beyond that level makes the participation in the voting less rewarding and this reward diminishes at an increasing rate.

The bottom point of the discussion at the moment would be the following. Fraud of the type discussed above can increase the participation rates. But this is true only if it is undertaken at an extent enough just to win the elections at the smallest margin (or around it). The strategic motives to vote diminish at high rate once the fraud goes beyond the level guaranteeing simple victory in binary elections.

Intervention that changes the voting costs distributions

Existence of certain voters supporting the Incumbent or the mere ballot-stuffing are not the only type of fraud. There are two other broad forms of interventions applied by the Incumbents: decrease the voting costs of its supporters or increase the voting costs of those favouring the Challenger. We will consider these two fraud mechanisms in an unified framework. There are a few important issues to be addressed. First question is about the possible efficiency of this fraud mechanism in achieving the required results (will it be the victory with a small or with a considerable margin). Second, and obviously related issue, is the relationship between the fraud mechanisms applied, whether the interventions types are complements or are quite different in the outcomes achieved.

[6]The expected number of agents, λ, in this simulation is $2,000$, $C = 0.08$ and the ex-ante support for the Challenger within the population is 0.54

Figure 3: Turnout rate as a function of the fraud level (number of certain voters)

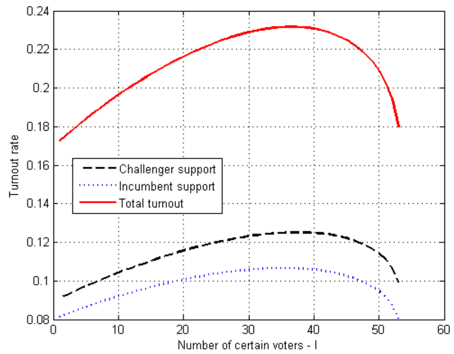
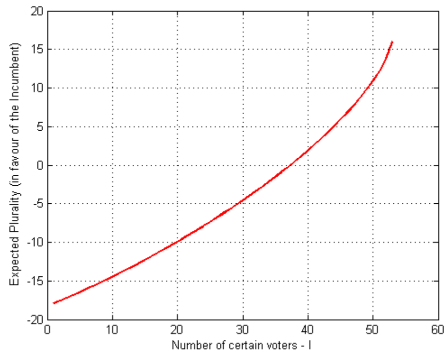


Figure 4: Expected plurality as a function of the fraud level (number of certain voters)



Finally, the main question is about the effect on the turnout rates and whether the logic observed with the "certain voters" goes through also with the cost related fraud mechanisms.

To continue our analysis we have to formulate the exact mechanism by which the voting costs of the voters are modified. Our approach is summarized in the assumption below.

Assumption 4. The voting costs of the agents supporting both the Incumbent and the Challenger are uniformly distributed. The voting costs distribution of the Incumbent supporters first order stochastically dominates the voting costs distribution of the Challenger supporters. Equivalently:

$$c_I \sim \mathcal{U}[0, \beta\mathcal{C}] \tag{26}$$

$$c_S \sim \mathcal{U}[0, \mathcal{C}] \tag{27}$$

where c_I and c_S are the voting cost distributions of the voters supporting the Incumbent and the Challenger correspondingly, whereas $\beta\epsilon\ (0, 1]$, as it was set out in the description of the model, is the coefficient by which the voting costs of the Incumbent's supporters are decreased.

This is the simplest cost modification technique which could have been applied. At the same time it captures the main idea of the intervention. It makes the voting on average less costly for the supporters of the Incumbent. The vote buying, transportation to the voting stations and any other type of in kind gifts to the voters can be attributed to the mechanism which we are modelling here.

Now the answer to the question whether creating cost asymmetries between the two groups of the voters is enough to change the outcome of the election is presented with the help of the following Proposition.

Proposition 3. If the fraudulent intervention creates ***only*** differences in the voting cost distributions between the groups supporting the Challenger and the Supporter in a way described in Assumption 4, then the extent of intervention enough to equalize the turnout rates for the two groups is defined by the ratio of ex-ante minority support to the ex-ante majority support:

$$s_C = s_B \text{ if } \beta^* = \frac{\gamma}{(1-\gamma)} \tag{28}$$

Consequently, if $\beta < \beta^*$, the Incumbent obtains larger share of the votes and the opposite is true.

Proof. See Appendix.

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Note the important difference between the two fraud mechanisms: with the "certain" voters of ballot stuffing the strategic turnout was always higher in the majority group and the winning margin was obtained directly from the fraudulent votes, whereas with the asymmetric voting costs distribution the required outcome is achieved as a result of the strategic voters response.

Remark 2. As long as the β is larger then the threshold level derived in Proposition 3 than it is true that the difference between the s_C and s_I is decreasing in β. Then directly from Proposition 2 it follows that the number of "certain voters" required to ensure the election outcome acceptable for the Incumbent will be lower than with $\beta = 1$ (no fraud causing differences in the voting costs).

The Remark 2 is based on the observation, that applying the Assumption 4 to the results obtained so far, the condition (25) proved to hold in the equilibrium is restated as follows:

$$\frac{s_C}{s_I} = (\beta\frac{1-\gamma}{\gamma})^{2/3}(\frac{I_l(2\lambda\sqrt{s_C s_I})}{I_{l+1}(2\lambda\sqrt{s_C s_I})})^{2/3} \lesseqgtr 1 \tag{29}$$

It is important to underline that up to now we have been implicitly assuming that the goal of the Incumbent is just to win the elections. Given the results obtained so far we are in a position to state that the two fraud schemes are in fact complements when it relates to the issue of expected plurality. The final decision which fraud type to apply depends on the cost functions of the both interventions. Depending on circumstances combination of the both types may be required. Of course the core issue is the Incumbent's intervention cost function.

To obtain a more complete picture of the two types fraud effect on the turnout rate and the expected plurality we repeat the simulations for various versions of parameter β (see appendix, Figures 5 and 6 for the results) combined with the first type intervention - certain voters.

Numerical analysis confirms that the two fraud mechanisms described are in fact complements both in terms of the impact on the turnout level and the election outcome. Higher is the difference between the voting costs the higher is the turnout and this is true up to the point when the participation rates within the two groups equalize. In essence the two different fraud mechanisms have the same impact on the turnout and the outcome, with only one important reservation. The turnout actually observed will be higher with the first ("certain" voters) approach if we include in our calculations of the turnout rates also these votes. So far we have been ignoring their impact on the turnout rate and we were focusing only on the strategic voters, but it is important to keep in mind that these voters (or these ballots) will be eventually included in the final turnout rate determination process.

3 Conclusion

We have investigated the fraud effect on the voter turnout categorizing the illicit interventions into two broad groups: fraud affecting the pivot probability and the fraud changing the voting costs distribution among the agents. To avoid complicated game involving the voters and the falsifying Incumbent we introduce a new parameter of the model representing the actual difference in votes that the Incumbent wants to ensure. As a result the extent of the fraudulent intervention is determined in the equilibrium outcome of the model.

So how successful was this attempt to address our questions of interest? Strategic models of costly voting indeed enable to analyse the behavioural response of the voters to the various types of fraudulent interventions. The model with incomplete information about the voting costs of the agents and only probabilistically known number of the voters enables considerably richer analysis. The types of equilibrium analysed for the complete information version of the model though conveys some interesting information doesn't provide clear predictions about the efficiency of intervention and the variations in the voter turnout rate. We have been able to demonstrate that if the fraud applied is of a magnitude to ensure for the falsifying candidate a victory at the margin, then the participation rates of the voters sustaining both alternatives are increased compared to the no fraud situation. These results hold independently of the type of the intervention applied - introduction of “certain” voters in favour of the falsifying Incumbent or the creation of differences in the voting costs distributions between the two groups of voters. At the same time, the strategic motives to vote diminish at high rate once the fraud goes beyond the level guaranteeing simple victory in binary elections.

Another important outcome of the current analysis is the estimation of the actual extent of intervention necessary to ensure the simple victory for the Incumbent. Numerical estimations prove that due to the strategic response by the voters the equilibrium volume of the intervention is higher than the level which would be required in case the Incumbent had been able to convince the voters in fully fair conduct of the forthcoming elections and apply ex-post correction of the unfavourable results. Note that this can be one of the reasons why the Incumbent forces in the developing countries characterized with non fully democratic elections try to persuade the electorate in their integrity and the intention to refrain from illicit activities.

The policy implication of the analysis is quite clear. Given that the elections do not fully conform to the democratic standards, the higher turnout rate, for instance compared to the previous election undertaken, cannot be considered as a factor supporting the argument about better organized elections.

4 Appendix

Proof of Proposition 1

To prove the proposition 1, we need to show that the following identities hold:

Lemma 2. Following identities hold:

$$\sum_{k=0}^{min[N_C-1,N_I-l]} \binom{N_C-1}{k+l}\binom{N_I-l}{k} = \binom{N_C+N-l-1}{N_I} \tag{30}$$

$$\sum_{k=0}^{min[N_C,N_I-l-1]} \binom{N_C}{k+l+1}\binom{N_I-l-1}{k} = \binom{N_C+N_I-l-1}{N_I} \tag{31}$$

Proof of Lemma 2:

To prove that the result is correct it is necessary to demonstrate that:

$$\sum_{k=0}^{min[N_C-1,N_I-l]} \binom{N_C-1}{k+l}\binom{N_I-l}{k} = \binom{N_C+N_I-l-1}{N_I}$$

Let us assume that $N_I < N_C$ and $l \geq 1 \Rightarrow N_I - l < N_C - 1$. So the RHS of the expression above could be re-written in expanded as follows:

$$\sum_{k=0}^{N_I-l} \frac{(N_C-1)!}{(k+l)!(N_C-1-k-l)!}\frac{(N_I-l)!}{k!(N_I-l-k)!} =$$
$$\sum_{k=0}^{N_I-l} \frac{(N_C-1)!}{(k)!(N_C-1-k-l)!}\frac{(N_I-l)!}{(k+l)!(N_I-l-k)!}$$

Also we know that by definition:

$$(N_I-l)! = \frac{N_I!}{\prod_{i=0}^{l-1}(N_I-i)} \tag{$*$}$$
$$(N_C-1)! = (N_C-1-l)!\prod_{i=0}^{l-1}(N_C-1-i) \tag{$**$}$$

Replacing the respective factorials with $(*)$ and $(**)$

$$\sum_{k=0}^{N_I-l} \frac{(N_C-1-l)!}{k!(N_C-1-k-l)!}\frac{N_I!}{(k+l)!(N_I-l-k)!}\prod_{i=0}^{l-1}[\frac{N_C-1-i}{N_I-i}] =$$

$$\sum_{k=0}^{N_I-l}\binom{N_C-1-l}{k}\binom{N_I}{k+l}\prod_{i=0}^{l-1}[\frac{N_C-1-i}{N_I-i}]=$$
$$\sum_{k=0}^{N_I-l}\binom{N_C-1-l}{k}\binom{N_I}{(N_I-l)-k}\prod_{i=0}^{l-1}[\frac{N_C-1-i}{N_I-i}]$$

Using the Vandermonde's identity:

$$\sum_{k=0}^{A}\binom{N_C}{k}\binom{N_I}{A-k}=\binom{N_C+N_I}{A}$$

where $A = min[N_C, N_I]$, the obtained expression can be re-written as:

$$\binom{N_C+N_I-1-l}{N_I-l}\prod_{i=0}^{l-1}[\frac{N_C-1-i}{N_I-i}]=\frac{(N_C+N_I-1-l)!}{(N_I-l)!(N_C-1)!}\prod_{i=0}^{l-1}[\frac{N_C-1-i}{N_I-i}]$$

Once again using the $(*)$ and $(**)$ instead of $(N_I-l)!$ and $(N_C-1)!$ we obtain:

$$\frac{(N_C+N_I-1-l)!}{\frac{N_I!}{\prod_{i=0}^{l-1}(N_I-i)}(N_C-1-l)!\prod_{i=0}^{l-1}(N_C-1-i)}\prod_{i=0}^{l-1}[\frac{N_C-1-i}{N_I-i}]=$$
$$\frac{(N_C+N_I-1-l)!}{N_I!(N_C-1-l)!}=\binom{N_C+N_I-1-l}{N_I}$$

The same result is achieved when we assume that $N_C \leq N_I$ as well as for the identity (31), so we skip the proof of that part.

End of Lemma 2 proof.

Statement (1) of Proposition 1 is obtained as a result of direct application of the Lemma 2 and using the equilibrium type condition $\sigma_C = 1 - \sigma_I$ to the equilibrium conditions (10) and (11).

Statement (2) is directly obtained by differentiating condition (12) w.r.t σ_C.

Statement (3) is the results of obtaining condition (12) (Statement 1). By following the procedure to obtain the condition (12) it is obtained that equilibrium conditions (10) and (11) coincide. As a result an equilibrium with different cost levels between the two groups of the voters is not achievable.

Statements (4) and (5) of the Proposition unfortunately do not have analytical proves and are obtained as a result of numerical estimations obtained so far and are summarized in the Figure 2.

End of Proposition 1 proof.

■

Proofs of Proposition 2 and 3

The first step is to define the probabilities of casting a decisive vote (pivot probabilities) used the Bayesian Nash equilibrium conditions (19) and (20). Given that the number of strategic voters has a Poisson distribution, the number of votes cast for the Challenger (C) and the Incumbent (I) follows the same distribution, but of course with different expected number of voters - $\tilde{N}_C \sim \mathcal{P}(\lambda_C)$ and $\tilde{N}_I \sim \mathcal{P}(\lambda_I)$. Then, using the definition of Poisson distribution, as well as taking into account the existence of "certain" (non-strategic) voters in favour of the Incumbent we obtain that pivot probabilities are given by:

$$prob[\tilde{n}_C = \tilde{n}_I + l] = \sum_{k=0}^{\infty} prob[\tilde{n}_C = k + l]prob[\tilde{n}_I = k] = \sum_{k=0}^{\infty} \frac{e^{-\lambda_C - \lambda_I} \lambda_C^{k+l} \lambda_I^k}{(k+l)!k!} \tag{32}$$

$$prob[\tilde{n}_C = \tilde{n}_I + l + 1] = \sum_{k=0}^{\infty} prob[\tilde{n}_C = k + l + 1]prob[\tilde{n}_I = k] = \sum_{k=0}^{\infty} \frac{e^{-\lambda_C - \lambda_I} \lambda_C^{k+l+1} \lambda_I^k}{(k+l+1)!k!} \tag{33}$$

Further, we use the modified Bessel function to present the conditions (32) and (33) in a more compact way. The modified Bessel function has the following form:

$$I_c(2z) = \sum_{k=0}^{\infty} \frac{z^{2k+c}}{(k+c)!k!}$$

So, using that:

$$\lambda_C^{k+l} = \lambda_C^{\frac{1}{2}(2k+l)} \lambda_C^{\frac{1}{2}l}$$

$$\lambda_I^{k} = \lambda_I^{\frac{1}{2}(2k+l)} \lambda_I^{-\frac{1}{2}l}$$

And by denoting $z = \sqrt{\lambda_C \lambda_I}$, $c = l$, we obtain that the pivot probability (32) can be re-written as follows (the same is applicable to pivot probability given in (33)):

$$prob[n_C^i = n_I + l] = e^{-\lambda_C - \lambda_I} [\frac{\lambda_C}{\lambda_I}]^{\frac{1}{2}l} \sum_{k=0}^{\infty} \frac{\sqrt{\lambda_C \lambda_I}^{2k+l}}{(k+l)!k!} = e^{-\lambda_C - \lambda_I} [\frac{\lambda_C}{\lambda_I}]^{\frac{1}{2}l} I_l(\sqrt{\lambda_C \lambda_I}) \tag{34}$$

Finally, denoting $s_C = \frac{\lambda_C}{\lambda}$, $s_I = \frac{\lambda_I}{\lambda}$, and by applying the definition of uniform distribution for the voting costs we obtain the conditions (21) and (22) of the Proposition 2.

For the Proposition 3 we take into account that $l = 0$ ("certain" voters or ballot stuffing are not used as a fraud technique). This enables us to apply the asymptotic property of the modified Bessel function above, which is given by:

$$\lim_{z\to\infty} I_c(z) = \frac{e^z}{\sqrt{2\pi z}}$$

Note that the asymptotic property of the Bessel function stated just above is valid in the case when c is considerably smaller than the z. In the Proposition 2 $c = l$ could be of comparable magnitude with the z ($z = \sqrt{\lambda_C \lambda_I}$), whereas for the Proposition 3 with $l = 0$ this constraint is no more binding. So, applying the asymptotic property of the modified Bessel function we obtain that:

$$s_C = \frac{(1-\gamma)}{\mathcal{C}} \frac{e^{-\lambda(s_C - s_I)^2}}{2\sqrt{\pi\lambda}\sqrt[4]{s_C s_I}} \tag{35}$$

$$s_I = \frac{\gamma}{\beta\mathcal{C}} \frac{e^{-\lambda(s_C - s_I)^2}}{2\sqrt{\pi\lambda}\sqrt[4]{s_C s_I}} [\frac{s_C}{s_I}]^{\frac{1}{2}} \tag{36}$$

Now dividing the equilibrium condition (35) by the condition (36) we directly obtain:

$$\frac{s_C}{s_I} = [\frac{1-\gamma}{\gamma}\beta]^{\frac{2}{3}} \tag{37}$$

From (37) the result of the Proposition 3 is obtained directly.

■

Figure 5: Turnout rate as a function of the fraud level

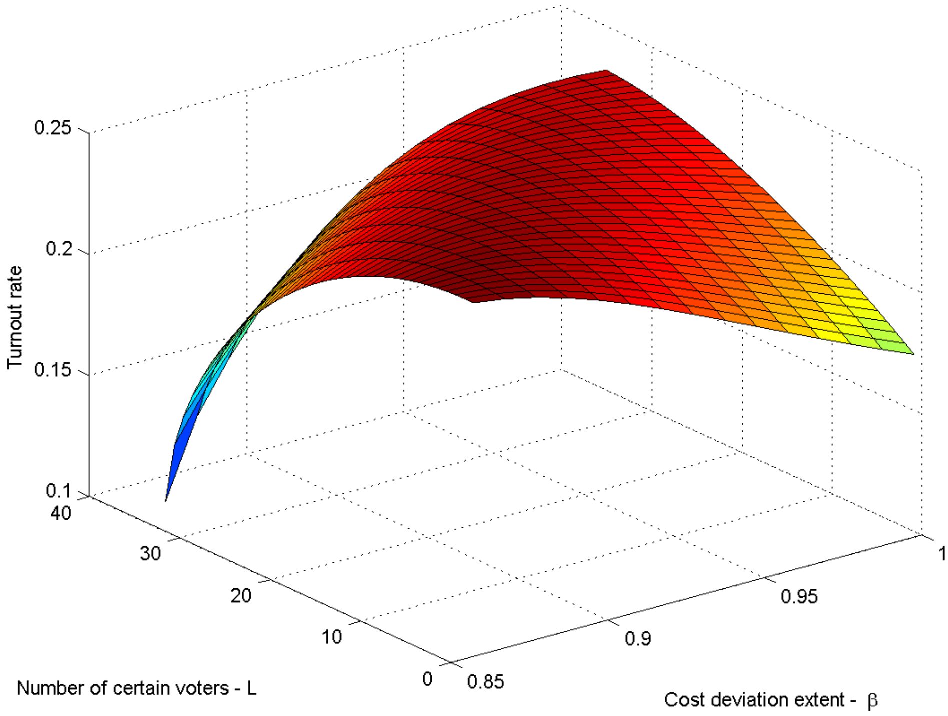
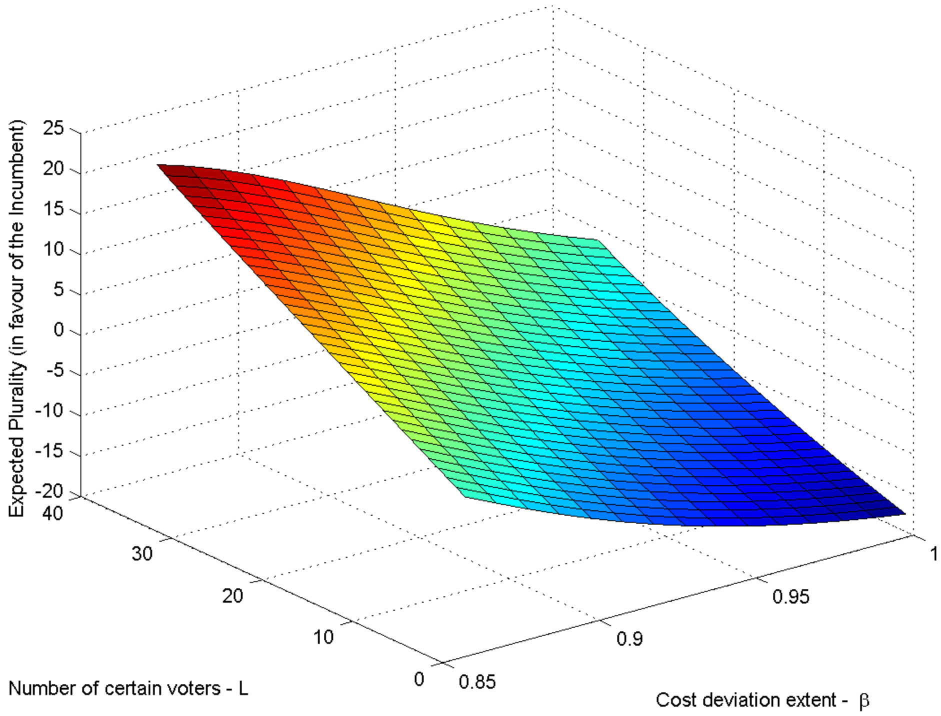


Figure 6: Expected plurality as a function of the fraud level



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