Proportional Systems with Free Entry. A Citizen-Candidate Model*
PRELIMINARY AND INCOMPLETE DO NOT QUOTE

Paolo Balduzzi§ Sandro Brusco¶

October 11, 2012.

Abstract

We analyze the equilibrium of a proportional electoral system with free entry in a citizen candidate model. In proportional systems the policy outcomes are typically decided through legislative bargaining. We consider two models of legislative bargaining. When policy is determined by a 'random formateur' chosen with probability proportional to the seats won, there are only two types of equilibria: dominant party equilibria in which one party gets the absolute majority or 'maximum entry' equilibria in which all parties are small. When policy is determined by forming a coalition with at least 50% of the seats then additional equilibria appear, where some parties are of intermediate size.

1 Introduction

It is part of the conventional wisdom that proportional systems tend to have a larger number of parties than majoritarian ones. However, exactly how many parties can we expect? And how large tend the parties to be in a proportional system? To answer these questions we have to consider a model of free entry in the electoral arena, which requires making assumption on what motivates would-be politicians.

The standard model of purely office–seeking politicians does not appear to offer interesting answers. Even if we ignore potential equilibrium existence problems, a

*Part of this paper was written while Brusco was visiting Università Cattolica in Milan. We thank participants to GRASS for helpful comments.

§Catholic University of Milan. E-mail: paolo.balduzzi@unicatt.it; skype: paolo.balduzzi.

¶Department of Economics, Stony Brook University. E-mail: sandro.brusco@stonybrook.edu.

model in which the only goal of parties and politician is to win seats leads to the prediction that, with a proportional system, there will be as many parties as seats available[1]. Thus, with free entry we should observe a large number of very small parties.

While it is surely true that many small parties obtain representation in proportional systems, it is also quite common to observe relatively large parties. For example, Taagepera and Laatsit [23] report that the average (seats) sizes of the top two parties running in 264 proportional elections ranges from 22.7% and 18.5% to 58.8% and 21.1%, with the number of remaining parties decreasing as the first gets bigger. Diermeier and Merlo [11] report that in 313 post war elections in 11 democratic countries the average and median number of represented parties is, respectively, 7.35 and 7. This is typically far below the maximum number of parties that can obtain representation. Further evidence that large parties are common in proportional systems is provided in Rimanelli [21]. A model of the proportional system should therefore be able to produce equilibria in which at least some of the parties are either medium-sized or large.

We consider a variant of the citizen-candidate model. A continuum of citizens with ideal points distributed on the interval $[0, 1]$ can run for a finite number of seats. Each citizen can form a party, whose platform is identified by the ideal point of the party-founding citizen, and run for election. Citizens pay a cost for running and get utility from winning seats and participating to government. Furthermore, by running and winning seats they get a chance to move the policy implemented closer to their ideal point. Since the number of seats is finite, in equilibrium only a finite set of parties enter.

Proportional elections do not necessarily lead to a clear winner. More often than not many parties obtain representation and the policy to be implemented is decided *ex post* through some legislative bargaining game. When candidates care about policy, the exact form of the legislative game chosen is important. We discuss two different models and explore their implication for the number and size of the party entering the electoral contest.

The simplest model is the 'random formateur' model analyzed in Diermeier and Merlo [11]. According to the model, a party is chosen to form the government with a probability that depends on its share of the vote. If the share is greater than

[1] This statement shold be qualified. We have in mind a model in which each party proposes a single political platform and all the candidates belonging to a party commit to implement that platform. In principle different candidates inside a party may try differentiate their policy platforms, something that may be particularly attractive under 'open list' systems. Our model is closer to a 'closed list' system in which the party leader has control over the seats won by the party.

50% the party is chosen with probability 1, while if no party has a majority of seats the probability of selecting party i is equal to the share of the seats of the party. Once a party becomes formateur it forms a government (we may think that minority governments are able to secure a parliamentary majority because enough representatives prefer avoiding a new election) and it obtains the government rents.

The more complicated model requires that, when no party has a majority of the seats, a coalition achieving an absolute majority be formed. We assume that the policy preferred by each party belonging to the coalition is chosen with probability equal to the share of seats of the party over the number of seats of the coalition.

It is intuitive that equilibria with maximum entry always exist, under both versions of the policy determination process. The question is whether other equilibria, with large or medium-sized parties, exist.

We show that under the 'random formateur' model there is only one type of equilibrium not involving maximum entry: a 'dominant party' equilibrium, in which one party gets an absolute majority of seats. The intuition is that citizens voting for the dominant party are reluctant to form new parties, even if such parties can collect a sizable number of seats, since this reduces the share of the dominant party below the 50% threshold, thus inducing a discontinuity in the policy choice: instead of implementing the policy preferred by the dominant party with probability 1, the policy is chosen through a lottery with probabilities determined by the seat shares. This discontinuity is crucial in keeping together the voters of the dominant party. If the party is large but not dominant then creating a new party always improves the policy implemented (from the point of view of the creator of the new party), thus no equilibrium with medium-sized party can exist.

Equilibria in which large but non-dominant parties are formed may exist when the formateur has to form a coalition in order to govern. The intuition is that citizens voting a large but not dominant party are unwilling to break away and form a different party because now they may be excluded more often from the governing coalitions which are formed. On average a breakway party will improve utility in terms of policy when it is included in a governing coalition, but this is countered by a lower probability of belonging to the governing coalition as well as by a worsening of the policies implemented when excluded. The net effect may be a decrease in the expected utility of the policies implemented. If this effect is large enough for all citizens who are excluded, it will more than balance any gain in utility obtained by winning seats. Of course this kind of equilibrium requires a particular configuration of medium sized parties. We show that there is a lower bound on the number of parties present in these equilibria: at least four parties must be present, and this lower bounds hold for all distributions on preferences.

We also show how are framework can be used to analyze variants of the proportional system, such as the introduction of thresholds and prizes.

[**DISCUSS RESULTS HERE**].

The rest of the paper is organized as follows. We start with a discussion of the existing literature in section 2. Section 3 contains the theoretical model and the results for the 'random formateur' case, while section 6 deals with the case in which coalitions have to be formed when no party obtains a majority of seats. We then consider in section 7 some variants of the proportional system, such as thresholds for representations and majority prizes. Section 8 discusses some possible extensions of the theory, including alternative behavioral assumptions, and section 9 contains the conclusions.

2 Literature

As previously mentioned, an important difficulty in modeling proportional systems is that elections may not yield a clear winner, so that the policy that ends up being implemented depends on the legislative bargaining game occurring after the election. Entry and positioning decisions at the electoral level thus depend on the expected outcome at the bargaining stage. As argued by Blais and Bodet [6], proportional systems tend on one hand to produce parties with political positions which are quite dispersed around the median voter, but on the other hand they tend to impose coalition governments and compromise solutions that bring the actual policies implemented closer to the median. Thus, whether the actual policies implemented are more or less 'centrist' than, say, in a plurality system is an empirical matter.

Austen-Smith and Banks [2], Austen-Smith [1] and Baron and Diermeier [4] have analyzed models of proportional representation but only with a fixed (usually three) number of parties. The main question which these models try to anwer is how likely it is that a coalition government be formed. These models are clearly not suitable for analyzing the question of how many parties will be formed in equilibrium when there is free entry and many different political positions may be held.

More recently some papers have analyzed models in which the number of parties is endogenous. The two papers which are more closely related to ours are Hamlin and Hjortlund [13] and Bandyopadhyay and Oak [3]. Hamlin and Hjortlund also consider a citizen-candidate model in which citizens can freely form parties and then seats are distributed proportionally. There are two important differences between their model and ours. First, we adopt a different rule for policy formation. Hamlin and Hjortlund assume that the policy implemented is the weighted average of the ideal points of the parties obtaining representation. We additionally assume that whenever a party obtains a majority of the seats then it can implement its ideal point. This rule is obviously more realistic and it has very important consequences in terms of strategic behavior. The second important difference is that we assume that political rents accrue to parties in proportion to the share of seats that they obtain, while Hamlin and Hjortlund assume that rents are obtained only by the party with the highest vote share. This also implies very different strategic incentives. Under our assumption incentives to entry are much stronger, in particular for parties who expect to receive a small share of the vote. This tendency to the proliferation of parties is widely considered an important characteristic of proportional representation[2].

The main difference between Bandyopadhyay and Oak [3] and our paper is that they assume that rents are linked to government participation, while we assume that rents are linked to parliamentary participation. To see how the two hypotheses may lead to very different results, observe that in the Bandyopadhyay and Oak model there is an equilibrium in which only the citizen located at the median enter. No other candidate is willing to enter because it cannot affect the policy outcome and, since it remains out of the government, no political rents are obtained. This is not possible in our model, as long as entry costs are sufficiently low. If there is a single party located at the median then a new party entering close to the median can get almost 50% of the vote. Our assumption that rents are linked to parliamentary representation implies that such an opportunity of creating a large party will not be overlooked. In section 8 we discuss the introduction of government rents *in addition* to parliamentary rents.

We are not aware of any work discussing the impact of variants of the proportional system, such as thresholds and prizes, from a theoretical point of view. There is however a theoretical literature on general models of electoral rules, including proportional rules, and a specific literature on pure proportional systems. Cox [8], [9] produces a general analysis for the case of office–seeking parties looking at the properties of equilibria under many different systems, allowing in particular for multi-member districts. His analysis however takes the number of parties as fixed and does not allow for free entry. This implies that the question 'how many viable parties does a certain electoral system induce' cannot be answered. Second, existence of equilibrium is not generally established and this makes some of the characterizations

[2]Another minor difference between our paper and Hamlin and Hjortlund [13] is that we assume a finite number of seats, so the share of seats may differ from the share of the vote. Also, our results are obtained for general distributions, while most of their results are obtained for the uniform distribution.

empty.

Myerson and Weber [18] also propose a general analysis of voting equilibria that can be applied to different institutions. They allow for strategic behavior on the part of the voters and define an equilibrium as a situation in which the expectations of the voters on the relative strength of the parties are correct. Voting equilibria always exist and the median voter theorem typically does not apply.

[MORE HERE].

3 The Model

A continuum of voters, with ideal points distributed on the interval $[0,1]$, have to elect a finite number M of representatives. There is a single electoral college and the electoral system is proportional[3]. Let $x \in [0,1]$ denote the preferred policies of the citizens and let $F(x)$ be the cumulative distribution function of the ideal points. We assume that F has a differentiable and strictly positive density $f(x)$.

Each citizen can form a party. We will follow Osborne and Slivinski [20] and assume that a citizen with ideal point x can only propose as party platform the implementation of policy x (alternatively, there is no way to commit to a policy, so everybody expects that a candidate with ideal point x will try to implement policy x). Citizens prefer policies closer to their ideal point. Furthermore, they pay a cost whenever they decide to form a party and they obtain political rents which are proportional to the share of seats they obtain in an election. We refer to the latter as *parliamentary rents*.

The strategy set of each citizen is simply $\Sigma = \{0,1\}$, where $\sigma = 0$ means that the citizen is not running and $\sigma = 1$ means that the citizens is forming a party and participating to the election. The utility function of a citizen located at x when a policy $\bar{x}$ is implemented and a share s_x of seats is obtained in case of participation to the election is given by

$$u_x(s_x, \bar{x}, \sigma) = \alpha(s_x - c)\sigma - |x - \bar{x}|,$$

where $c > 0$ is the cost of running. The parameter α reflects the relative weight that the citizen assigns to the parliamentary rents with respect to its policy preferences. We will maintain the following assumption on the parameters.

Assumption 1 *$\alpha > 0$ and $0 \leq c < 1$.*

[3]There are many different types of proportional systems (e.g. d'Hondt, Hare, etc.). For the moment we leave this unspecified, but the issue will be discussed later.

The assumption implies that, in equilibrium, there is always at least one party running. Notice also that a party will always run if $c = 0$, the policy implemented after the election is unaffected by its presence and there is a possibility of winning at least one seat. As α goes to zero citizens are only motivated by policy and parliamentary rents become irrelevant, while as α goes to $+\infty$ we have a model in which parties care almost exclusively about parliamentary rents.[4]

Standard citizen-candidate models of the first-past-the-post system assume that the policy implemented is the one proposed by the top vote-getter. This is not necessarily the case in a proportional system. Unless a party gets an absolute majority of the seats, the policy will be determined by some kind of compromise agreement among different parties. Thus, in this case, it becomes crucial to model the legislative bargaining game that takes place after the election. We adopt the 'random formateur' model proposed by Baron and Diermeier [4], which works as follows. Whenever a party gets the absolute majority of seats then that party forms the government. When no party has the majority of the seats then each party has a probability of becoming the formateur equal to its share of the seats. More precisely, let $\lfloor x \rfloor$ be the integer part of x. Then a party becomes formateur with probability 1 if the number of seats m_i assigned to that party is such that $m_i > \lfloor M/2 \rfloor$. Otherwise, the party becomes the formateur with probability $s_i = m_i/M$. Notice that the function describing the probability of being a formateur has a 'discontinuity' at $s_i = \frac{1}{2}$, where s_i is the share of seats won by party i. For example, a party obtaining 49% of the seats will get to form the government only with probability 49%, while a party obtaining 50% will form the government with probability 1. As it will become clear later, this discontinuity plays an important role in defining the set of equilibria.

We will consider a political game composed of three stages and we will look for subgame perfect equilibria of the game.

1. **The party formation stage.** At this stage each citizen decides whether or not to form a party and run. Each party is characterized by a political platform, given by the preferred point of the citizen founding the party.

2. **The electoral stage.** Let $\mathcal{N}$ be the set of parties entering at stage 1. If $\mathcal{N} = \emptyset$ (no party runs) that a status quo is implemented giving to each citizen a utility $\underline{u}_x \leq -1$. Otherwise, each citizen votes sincerely for the closest party. Let $v = \{v_i\}_{i \in \mathcal{N}}$ be the vector of vote shares, i.e. v_i is the share of the popular vote

[4] It is worth pointing out here that, even with $\alpha \to +\infty$, the model remains different from the standard Downsian model in which the parties care only about winning. The difference is that a citizen, when forming a party, is not free to choose the vote-maximizing platform. Instead, the platform has to coincide with the citizen's ideal point.

of party i. For each v, the seats assigned to each party are determined according to the electoral formula (Hare or D'Hondt). We will maintain the assumption that a party collecting a vote share v_i such that $v_i > m/M$ must be assigned at least m seats. Thus it must always be the case that $(m_i + 1)/M \geq v_i \geq m_i/M$.

3. **The legislative bargaining stage.** Let $\mathcal{J} = \{1, \ldots, J\}$ be the set of parties obtaining parliamentary representation given the vector of vote shares $v = \{v_i\}_{i \in \mathcal{N}}$. $\mathcal{J}$ is a (possibly proper) subset of $\mathcal{N}$. Since there is a finite number of seats, the set $\mathcal{J}$ is also finite. Call s_j the share of parliamentary seats of party j and $s = \{s_j\}_{j \in \mathcal{J}}$ the vector of seats. Also, define $s^{(1)} = \max_{j \in \mathcal{J}} s_j$ the seat share of the most voted party. We assume that party i is chosen as formateur with probability $p_i(s)$ defined as follows

$$p_i(s) = \begin{cases} 1 & \text{if } s^{(1)} > \frac{1}{2} \quad \text{and} \quad s_i = s^{(1)} \\ 0 & \text{if } s^{(1)} > \frac{1}{2} \quad \text{and} \quad s_i \neq s^{(1)} \\ s_i & \qquad \text{if} \quad s^{(1)} \leq \frac{1}{2}. \end{cases} \tag{1}$$

   Once a formateur j is chosen a legislative game $G_j(s, \mathcal{J})$ is played and the policy chosen is determined. We will use two different versions of the game $G_j(s, \mathcal{J})$, one in which the formateur can implement the preferred policy and another in which it has to form a coalition.

The assumption that a party with the absolute majority of seats is put in charge of government with probability 1 is completely natural and it does not really need to be defended. The rule that we adopt when no party has the majority of seats is not natural. In many parliamentary regimes, when there is no party with a clear majority the formateur is chosen discretionally by some higher authority (e.g. the President of the Republic), usually considering the probability that a certain politician may able to form a majority coalition. Assuming the formateur is chosen according to the function $p_i(s)$ should be seen as a simplified representation of such more complicated decision rules.

The two variants of the legislative game that we are going to consider are the following.

The simplest version is a *random dictator game*. The formateur, once chosen, is free to implement its preferred policy.

The more complicated version requires the formateur to form a coalition of parties reaching 50% of seats. We will call this the *random coalition maker game*, and we postone a precise description of the game to section 6. In short, we assume that the policy preferred by each party belonging to the coalition is chosen with probability given by the number of seats held by the party divided by the total number of seats controlled by the coalition.

4 General Results

In this section we present some results which do not depend on the particular form chosen for the legislative game. Let

$$\widehat{s}_M = \begin{cases} \left(\left\lfloor \frac{M}{2} \right\rfloor - 1\right)/M & \text{if } M \text{ even} \\ \left\lfloor \frac{M}{2} \right\rfloor/M & \text{if } M \text{ odd} \end{cases}$$

be the seat share of a party which has exactly one seat less than needed to get the absolute majority. The expression is equal to $\frac{1}{2} - \frac{1}{M}$ when M is even and to $\frac{1}{2} - \frac{0.5}{M}$ when M is odd. The next proposition collects some preliminary results which are independent of the particular variant of the proportional system adopted or the form of the legislative game.

Proposition 1 *There is no equilibrium in which no party runs. In a parliament with M seats, if the cost of entry is $c < \widehat{s}_M$ then there are no single party equilibria. If the cost of entry is $c \geq \widehat{s}_M$ there is always an equilibrium with a single party located at the median.*

Essentially, the proposition states that unless the cost of entering is prohibitive there must be multiple parties that enter. When there is a single party then an entrant can win a large share of the seats, and with a low cost of c this is enough to convince it to enter.

Since the case with a high c is somewhat obvious, we will dedicate most of the paper to discuss the case of a small c. This should also be seen as the most realistic case, since $c < \widehat{s}_M$ essentially says that the cost of running is no more than the benefit of obtaining almost 50% of the seats. In fact, we will assume $c = 0$ for most of the paper and discuss the effect of a positive c in Section 8.

The first question that we are going to ask is whether it is possible to have equilibria in which a single party obtains an absolute majority of seats. Let's say that an equilibrium has a *dominant party* if the party controls at least $\left\lfloor \frac{M}{2} \right\rfloor + 1$ seats. Situations in which a party obtains more than 50% of the seats in a proportional system are not frequent, but neither they are unheard of[5].

[5] Examples include the Social Democrats in Sweden in 1968, the CDU-CSU (taken as a single party) in Germany in 1957, the People's Party in Spain in 1996 and the Socialist Party in Spain in

In the next proposition we first characterize equilibria with a dominant party. The characterization holds both for the 'random dictator' and for the 'random coalition maker' models.

Proposition 2 *Suppose $c = 0$. Let $\{x_1, \ldots, x_J\}$ be an equilibrium configuration such that the party located at x_k is dominant. Then the dominant party wins at most $\lfloor \frac{M}{2} \rfloor + 2$ seats and all parties with index strictly greater than $k+1$ or strictly smaller than $k-1$ have at most two seats.*

If a party contiguous to the dominant one (i.e. a party located at x_{k-1} or x_{k+1}) has strictly more than two seats then the dominant party obtains a share of the vote which is barely enough to get $\lfloor \frac{M}{2} \rfloor + 1$ seats.

The intuition for the result is the following. Suppose that there is an equilibrium in which parties positioned at $\{x_1, \ldots, x_J\}$ enter and a dominant party gets a number of seats strictly larger than $\lfloor \frac{M}{2} \rfloor + 2$. Consider a citizen voting the dominant party, but with a different ideal point. In equilibrium this citizen must prefer to stay out rather than forming a new party. When $c = 0$ the main reason to stay out is the discontinuity in the policy function. If entry deprives the DP of the majority of seats then the policy selected becomes stochastic and this makes the entrant worse off. However, if the DP has a large majority of seats then there will be potential candidates at points where entry takes away some seats from the DP but not enough to deprive it of the majority. Since entry does not change the policy and gives to the entering party some seats, the original configuration cannot be an equilibrium. Thus, in equilibrium it must be the case that no entry of this type is profitable and this can only occur when the DP has a bare majority of seats.

Equilibria with a dominant party exist when the private benefits for holding seats are not much more important than policy consideration. Citizens voting for the dominant party face a trade off: by creating their own party they may get a sizable representation in the Parliament, but this deteriorates the probability distribution on policies, since it is no longer true that the policy preferred by the dominant party is implemented with probability one. Thus, we expect that dominant party equilibria exist when the private benefits from representation are small. The next proposition makes this intuition precise.

Proposition 3 ***TO BE COMPLETED***. *For any given distribution F there is an $\underline{\alpha}_F > 0$ such that for each $\alpha \in (0, \underline{\alpha}_F)$ an equilibrium with a dominant party exists. If $\alpha > 2$ then there is no dominant party equilibrium.*

1982 and 1986.

As it is frequently the case, the electoral game we are considering typically has many equilibria.

5 Random Dictators

However, it turns out that, when $c = 0$ and the value of α is 'reasonable' there are only two types of equilibria. Either there is a dominant party, or there is a multiplicity of very small parties, each one of them getting one or two seats. This is made precise in the next proposition.

Proposition 4 ***TO BE COMPLETED****. Suppose $c = 0$ and $\alpha \geq \frac{1}{M}$. If the equilibrium has no dominant party then each party gets at most two seats. No party is located at 0 or 1.*

Propositions 3 and 4 characterize the set of equilibria for 'reasonable' values of α. Basically either there is a dominant party or there are many small parties. Notice that there may be many equilibria. For example, in the class of equilibria with a dominant party there may be many positions that the dominant parties may occupy. The same holds true for the positions occupied by the parties in an equilibrium with only small parties.

6 Random Coalition Makers

In this section we consider a more complicated version of the legislative game $G_j(s, \mathcal{J})$ to be played after j has been named formateur. Specifically, we assume that the formateur chooses a subset $\mathcal{H} \subset \mathcal{J}$ of parties and proposes the formation of a coalition government. The subset $\mathcal{H}$ has to be such that $\sum_{j \in \mathcal{H}} s_j > \frac{1}{2}$

[MORE HERE].

The chosen parties sequentially[6] say 'accept' or 'no'. If all the parties in $\mathcal{H}$ accept and $\sum_{j \in \mathcal{H}} s_j > \frac{1}{2}$ then a government is formed, otherwise all citizens receive a utility $\underline{u}_x \leq -1$. The policy implemented is a probability distribution over the ideal points of the parties in $\mathcal{H}$. Defining $s_{\mathcal{H}} = \sum_{j \in \mathcal{H}} s_j$, the probability of selecting the ideal point x_i of party $i \in \mathcal{H}$ is $q_i^{\mathcal{H}} = s_i / s_{\mathcal{H}}$. Thus, only parties within $\mathcal{H}$ can determine the policy, while parties outside the governing coalition have no influence at all.

The outcome of this simple legislative game is as follows. Once party i has been selected as formateur, it will consider all possible governing coalitions and it will make

[6] We assume that parties accept sequentially in order to avoid irrelevant multiple equilibria that may occur when the parties act simultaneously.

a proposal to the coalition generating the highest expected utility. The members of the coalition will say yes. This is obviously a very crude model of legislative bargaining. It exhibits however the important feature that a party, if it has any hope of influencing policy, has to worry about the coalitions that it will be able to form. Notice that the assumption that the policy is determined according to a probability distribution over policy positions is equivalent to assuming that the parties cannot bargain *ex ante* over the policy to be implemented. If commitment to a policy were possible, then most legislative bargaining games would result in the implementation of the point preferred by the median party. We are essentially assuming that commitment is not possible, so that the policy is influenced *ex post* by the composition of the coalition.

Our conjecture is that now equilibria with medium-sized parties, i.e. parties which are not dominant but get a substantial number of seats, are possible. The main issue here is to explain how, absent the discontinuity related to dominant parties, a medium–sized party can avoid splitting into many smaller parties. In the previous case this was not possible. Here things are different.

Consider a citizen voting for a medium–sized party who is thinking to split the party. Forming a new party generates the following trade–off:

- on the plus side, your preferred point will be considered whenever the governing coalition includes your party;
- on the minus side, the 'more moderate' party that you are leaving is now more likely to form governing coalitions with parties which are far from you.

If the second effect is sufficiently strong then in equilibrium medium–sized parties can survive.

Another possible result: dominant party equilibria are more difficult, meaning that whenever a dominant party equilibrium exists under the coalition regime, it also exists under the random dictator regime. The reason is that the damage from losing majority by a close party is less, as a deviator at the center will be part of every coalition in which the former dominant party was excluded.

Suppose that three parties enter in equilibrium at the locations $\{x_1, x_2, x_3\}$ and that no one of them is dominant. Notice that this implies that any two-party coalition must have a majority of seats. Thus, no three–party coalition is formed. There are three possible coalitions, $\{x_1, x_2\}$, $\{x_2, x_3\}$ and $\{x_1, x_3\}$.

Lemma 1 *When there are only legislative rents there is no 3-party equilibrium such that* $s_i < \left\lfloor \frac{M}{2} \right\rfloor / M$ *.*

Proof. Let s_1, s_2 and s_3 be the seat shares of the 3 parties and assume that $s_i < \frac{1}{2}$ for each i.

Suppose first that whenever y enters just to the right of x_1 then the new seat shares distribution $\left(s_1', s_y', s_2', s_3'\right)$ is such that $s_y' = 0$. In that case a party could enter just to the left of x_1 and effectively replace x_1, since in the new distribution of seat share we would have $\left(s_y', s_1', s_2', s_3'\right)$ and $s_1' = 0$. Thus that cannot be an equilibrium unless by substracting the vote share $F\left(\frac{x_1+x_2}{2}\right) - F(x_1)$ it becomes possible to create other equilibria. Since $F\left(\frac{x_1+x_2}{2}\right) - F(x_1) < \frac{1}{M}$ (otherwise entry right to the left of x_1 would deliver at least one seat) the change cannot involve more than one seat. But if other equilibria become possible, that can only be because one party obtains one seat more than absolute majority.

What happens to the equilibria? Clearly y obtains utility from the extra seats. The probability that the prime minister is at $\{x_1, y\}$ remains the same (or slightly increases).

If x_1 is chosen then forming whatever coalition it was forming before plus y gives exactly the same utility. Excluding y, assuming it is possible, can only decrease utility because it leads to more weight on policy positions which are far away from x_1.

If x_2 is chosen then either it was making an offer to x_3 and noting changes or it was making an offer to x_1 and in that case it still prefer making an offer to $\{x_1, y\}$. Thus, y is not worse off.

If x_3 is chosen then either it was making an offer to x_2 and noting changes or it was making an offer to x_1 and in that case it still prefer making an offer to $\{x_1, y\}$.

Thus, y is never worse off in terms of the policy chosen and it is strictly better off because it has legislative rents. ■

Suppose that three parties enter in equilibrium at the locations $\{x_1, x_2, x_3\}$ and that no one of them is dominant. Notice that this implies that any two-party coalition must have a majority of seats. Thus, no three–party coalition is formed. There are three possible coalitions, $\{x_1, x_2\}$, $\{x_2, x_3\}$ and $\{x_1, x_3\}$.

Lemma 2 ***[INCOMPLETE]*** *When there are only legislative rents there is no 4-party equilibrium such that* $s_i < \left\lfloor \frac{M}{2} \right\rfloor / M$ *.*

Proof. Let s_1, s_2, s_3 and s_4 be the seat shares of the 4 parties and assume that $s_i < \frac{1}{2}$ for each i. Assume that the only coalitions which are formed are $\{x_1, x_2\}$ and

$\{x_3, x_4\}$. Entry in $y \in (0, x_1)$ should be discouraged by the fact that x_2 then decides to ally to x_3, excluding y.

Entry in (x_1, x_2) can only be discouraged by a change of preferred coalition by x_2. However, if y is just to the left of x_2 then such a change is not possible. Thus, there is always profitable entry. ■

The second result considers situations in which only contiguous coalitions are possible.

Lemma 3 *Suppose that the equilibrium has parties located at $\{x_1, \ldots, x_n\}$, no dominant party and that only contiguous coalitions are allowed. Then x_1 must be included in coalitions proposed by other parties or $x_1 < \frac{1}{M}$, and either x_n is included in coalitions proposed by other parties or $x_n > \frac{M-1}{M}$.*

Proof. Suppose that $x_1 > \frac{1}{M}$ and x_1 participates to a governing coalition only when x_1 is picked as formateur. Consider entry at $y = x_1 - \varepsilon$ for ε arbitrarily small. Then the party positioned at y obtains at least one seat. Thus, entry is unprofitable only if it change adversely the policy outcome. Since the sum of the seats of y and x_1 is the same, the parties located at $\{x_2, \ldots, x_{n1}\}$ have the same vote share as before entry. Since each one of them was forming a coalition government without x_1, whatever they were doing before entry by y is still feasible. Surely they do not form coalitions including both y and x_1, since that would be equivalent to forming a coalition with x_1 before entry and that was not done. Thus, any change must include the party at x_1. Since all the coalitions are contiguous can only happens if x_1 substitutes some party x_k on the far right of the coalition or x_1 is added to the coalition. In both cases the utility of y is strictly increased. We conclude that there is $\varepsilon > 0$ such that entry at $y = x_1 - \varepsilon$ is profitable, thus contradicting the fact that the original configuration is an equilibrium. The reasoning for x_n is symmetric. ■

The result shows that either a party is very extreme or it must be included in coalitions other than the ones proposed by itself.

Consider the the uniform distribution on the interval $[0, 1]$. We want to show that there is an equilibrium with 7 parties located at the positions $x_i = \frac{2i-1}{14}$. Thus, each party x_i gets $\frac{1}{7}$% of the votes and of the seats.

Suppose there is entry at $y < \frac{1}{14}$.

If x_2 forms the coalition $\{y, x_1, x_2, x_3, x_4\}$ utility is

$$-\frac{\left(\frac{y+\frac{1}{14}}{2}\left(\frac{3}{14} - y\right) + \left(\frac{1}{7} - \left(\frac{y+\frac{1}{14}}{2}\right)\right)\left(\frac{3}{14} - \frac{1}{14}\right) + \frac{1}{7}\left(\frac{5}{14} - \frac{3}{14}\right) + \frac{1}{7}\left(\frac{1}{2} - \frac{3}{14}\right)\right)}{\frac{4}{7}}$$

If x_2 forms the coalition $\{x_1, x_2, x_3, x_4\}$ utility is

$$-\frac{\left(\left(\frac{1}{7}-\left(\frac{y+\frac{1}{14}}{2}\right)\right)\left(\frac{3}{14}-\frac{1}{14}\right)+\frac{1}{7}\left(\frac{5}{14}-\frac{3}{14}\right)+\frac{1}{7}\left(\frac{1}{2}-\frac{3}{14}\right)\right)}{\frac{4}{7}-\frac{y+\frac{1}{14}}{2}}$$

$: \frac{1}{196} : \frac{1}{28} :$, Solution is: $\left(-\frac{1}{14}, \frac{1}{14}\right)$

Thus, x_2 prefers the coalition $\{x_1, x_2, x_3, x_4, x_5\}$

when

$$-20\frac{y-80}{y-120} > \frac{1}{100}y^2 - 16$$

which is always satisfied on the interval $(0, 20)$. Solution is: $\left(-10\sqrt{41}+50, 20\right) \cup \left(10\sqrt{41}+50, 120\right) : -14.031$.

Consider

We have to prove first that each one of the existing parties is better off not dropping out, and second that no profitable deviation is possible. Since the positions are symmetric it is sufficient to consider positions in the interval $[0, 50]$.

Consider x_1. If it drops out then it is clearly worse off in terms of policy: x_2 will select $\{x_2, x_3\}$, which is worse that $\{x_1, x_2, x_3\}$;

If x_3 selects $\{x_2, x_3\}$ the expected utility is

$$-\left(\frac{40}{60}\right)(50-30) = -\frac{40}{3}$$

If x_3 selects $\{x_2, x_3, x_4\}$ the expected utility is

$$-\left(\frac{40}{80}\right)(50-30) - \frac{20}{80}(70-30) = -20$$

If x_3 selects $\{x_3, x_4, x_5\}$ the expected utility is

$$-\left(\frac{20}{60}\right)(70-50) - \frac{20}{60}(90-50) = -20$$

Thus the selection is $\{x_2, x_3\}$ which is better, in terms of policy, than $\{x_2, x_3, x_4\}$.

The choice will be the same if x_4 and x_5 are selected as formateurs. Thus, when staying the expected utility is

$$-0.4\left(\frac{1}{3}(30-10)+\frac{1}{3}(50-10)\right)-0.2\left(\frac{1}{3}(30-10)+\frac{1}{3}(50-10)+\frac{1}{3}(70-10)\right) = -16$$

when dropping out the expected utility is

$$-0.6\left(\frac{3}{3}(30-10)+\frac{1}{3}(50-10)\right)=-20$$

so that he is better off staying.

Consider now x_2. By dropping out it is worse off when the formateur is x_1, since he will form the coalition $\{x_1, x_3\}$ which is worse than $\{x_1, x_2, x_3\}$. x_3 gets now 30% and its choice is $\{x_3, x_4\}$, so x_2 is again worse off.x_4 and x_5 must choose $\{x_3, x_4, x_5\}$, and x_2 in this case is better off because the weight given to x_3 is higher.

Thus, the expected utility for x_2 when he drops out is:

$$-0.3(30-10)-0.3\left(\frac{3}{5}(50-30)+\frac{2}{5}(70-30)\right)-0.4\left(\frac{3}{7}(50-30)+\frac{2}{7}(70-30)+\frac{2}{7}(90-30)\right)=$$
$$=-29.257$$

The expected utility when not dropping out is

$$-0.4\left(\frac{1}{3}(30-10)+\frac{1}{3}(50-30)\right)-0.2\left(\frac{1}{3}(50-30)+\frac{1}{3}(70-30)\right)+$$
$$-0.4\left(\frac{1}{3}(50-30)+\frac{1}{3}(70-30)+\frac{1}{3}(90-30)\right)=-25.333$$

which is higher.

Finally, consider x_3. He is worse woff in every possible case. Thus, it is a best response for each party to stay in.

We have now to consider possible entry.

Consider entry in the interval $[0, 10)$. Then y gets a share $\frac{10+y}{2}\leq 10\%$ of the vote. The expected utility of y under no entry is

$$-0.4\left(\frac{1}{3}(10-y)+\frac{1}{3}(30-y)+\frac{1}{3}(50-y)\right)-0.2\left(\frac{1}{3}(30-y)+\frac{1}{3}(50-y)+\frac{1}{3}(70-y)\right)+$$
$$-0.4\left(\frac{1}{3}(90-y)+\frac{1}{3}(50-y)+\frac{1}{3}(70-y)\right)=y-50$$

To compute the expected utility of y under entry we check for the various outcomes

- If y is chosen (probability $\frac{10+y}{200}$) then the coalition is $\{y, x_1, x_2, x_3\}$. The utility is

$$-\left(\frac{20-\frac{10+y}{2}}{60}(10-y)+\frac{1}{3}(30-y)+\frac{1}{3}(50-y)\right)=-\frac{1}{120}y^2+y-\frac{175}{6}$$

- If x_1 is chosen the coalition can be either $\{y, x_1, x_2, x_3\}$ or $\{x_1, x_2, x_3\}$. In the first case the utility of x_1 is

$$-\left(\frac{\frac{10+y}{2}}{60}(10-y)+\frac{1}{3}(30-10)+\frac{1}{3}(50-10)\right)=\frac{1}{120}y^2-\frac{125}{6}$$

In the second case utility is

$$-\left(\frac{20}{60-\frac{10+y}{2}}(30-10)+\frac{20}{60-\frac{10+y}{2}}(50-10)\right)=\frac{2400}{y-110}$$

thus x_1 prefers $\{y, x_1, x_2, x_3\}$. The expected utility of y is as above

- If x_2 is chosen the coalition can be either $\{y, x_1, x_2, x_3\}$ or $\{x_1, x_2, x_3\}$. In the first case the utility of x_2 is

$$-\left(\frac{\frac{10+y}{2}}{60}(30-y)+\frac{1}{3}(30-10)+\frac{1}{3}(50-30)\right)=\frac{1}{120}y^2-\frac{1}{6}y-\frac{95}{6}$$

In the second case utility is

$$-\left(\frac{20}{60-\frac{10+y}{2}}(30-10)+\frac{20}{60-\frac{10+y}{2}}(50-30)\right)=\frac{1600}{y-110}$$

it can be checked that x_2 prefers $\{x_1, x_2, x_3\}$. The expected utility of y is

$$-\left(\frac{20-\frac{10+y}{2}}{60-\frac{10+y}{2}}(10-y)+\frac{20}{60-\frac{10+y}{2}}(30-y)+\frac{20}{60-\frac{10+y}{2}}(50-y)\right)=\frac{1}{y-110}\left(y^2-120y+3500\right)$$

- In all other cases coalitions chosen are unchanged.

Thus, y prefers to stay out if

$$-0.4\left(\frac{1}{3}(10-y)+\frac{1}{3}(30-y)+\frac{1}{3}(50-y)\right)\geq$$
$$-0.2\left(\frac{20-\frac{10+y}{2}}{60}(10-y)+\frac{1}{3}(30-y)+\frac{1}{3}(50-y)\right)+$$
$$-0.2\left(\frac{20-\frac{10+y}{2}}{60-\frac{10+y}{2}}(10-y)+\frac{20}{60-\frac{10+y}{2}}(30-y)+\frac{20}{60-\frac{10+y}{2}}(50-y)\right)$$

, Solution is: $[-10.000, 110.0) \cup [130.0, \infty) \cup (-\infty, -10.0]$. So the inequality is always satisfied.

Notice what is the logic. Entry by y improves the situation when x_1 would have been selected, bu it worsen the situation when x_2, x_3 are selected because they now exclude the extremist party.

Consider now entry in the interval $(10, 30)$. In that case the party at y gets 10% of the vote. x_1 gets $\frac{10+y}{2}$ and x_2 gets $40 - \frac{30+y}{2}$. All other parties get 20%.

- If x_1 and y are chosen then the coalition is $\{x_1, y\ x_2, x_3\}$.
- Consider x_2. The two possible options are $\{x_1, y\ x_2, x_3\}$ and $\{y\ x_2, x_3, x_4\}$. Since x_4 is more far away from x_1 and it has a larger weight, x_2 prefers $\{x_1, y\ x_2, x_3\}$.
- Consider x_3. The coalition $\{x_2, x_3, x_4\}$ has more than 50%. This coalition is worse than in the original equilibrium since x_2 has less weight.
- the situation is unchanged under x_4 and x_5.

The utility of y before entry is

$$-0.4\left(\frac{1}{3}(y-10)+\frac{1}{3}(30-y)+\frac{1}{3}(50-y)\right)-0.2\left(\frac{1}{3}(30-y)+\frac{1}{3}(50-y)+\frac{1}{3}(70-y)\right)=$$
$$=0.333\,33y-19.333$$

$$-0.4\left(\frac{1}{3}(20-10)+\frac{1}{3}(30-20)+\frac{1}{3}(50-20)\right)-0.2\left(\frac{1}{3}(30-20)+\frac{1}{3}(50-20)+\frac{1}{3}(70-20)\right)$$

: -12.667The utility after entry is

$$\left(\frac{\frac{10+20}{2}}{60}(20-10)+\frac{40-\frac{30+20}{2}}{60}(30-20)+\frac{1}{3}(50-20)\right)+$$
$$-0.2\left(\frac{40-\frac{30+20}{2}}{80-\frac{30+20}{2}}(30-20)+\frac{20}{80-\frac{30+20}{2}}(50-20)+\frac{20}{80-\frac{30+20}{2}}(70-20)\right)$$

: -12.364

$$-0.4\left(\frac{1}{60}y^2-y+\frac{85}{3}\right)-0.2\left(\frac{1}{130-y}\left(y^2-160y+6300\right)\right)$$

$$0.333\,33y-19.333>-0.4\left(\frac{1}{60}y^2-y+\frac{85}{3}\right)-0.2\left(\frac{1}{130-y}\left(y^2-160y+6300\right)\right)$$

, Solution is: $(135.68,\infty)\cup(-\infty,10.003)\cup(24.320,130.0)$

With no entry

$$-0.2\,(x_3 + x_4 + x_5 - x_1 - x_2 - y)$$

With entry

7 Thresholds and Prizes

A common variant of the propositional system is obtained when thresholds for representation are imposed, i.e. a party can obtain seats only if it obtains at least a certain fraction of votes. It seems obvious that such thresholds should reduce the number of parties in equilibrium. It turns out that things are in fact a little bit more complicated.

Proposition 5 *Suppose $c = 0$ and a threshold equal to v^* is imposed for representation, i.e. only parties receiving at least a percentage v^* are assigned seats. Then:*

1. *Any equilibrium of the pure proportional system (i.e., without threshold) in which each party obtains a share greater or equal to v^* remains an equilibrium.*
2. *Assume that $\frac{1}{v}$ is an integer number. Then there is always an equilibrium in which exactly $\frac{1}{v}$ parties enter.*

The logic is straightforward. Suppose, for example, that there are M seats and a threshold of $\frac{1}{M}$ is imposed for representation. Then there is one equilibrium (there may be others with fewer parties) in which exactly M parties enter, located in such a way that each party gets exactly a share $\frac{1}{M}$. At that point no further entry is possible, since an entrant would get strictly less than $\frac{1}{M}$ and thus no representation.

One interesting and counter-intuitive implication is the following. Suppose that the conditions of Proposition 4 are not satisfied, so that a pure proportional system has only equilibria with fewer than M parties. Then imposing a threshold $\frac{1}{M}$, an apparently harmless modification, may actually *increase* the number of parties. This will occur if, after the introduction of the threshold, the citizens play the equilibrium described in Proposition 5.

More in general, suppose that under a pure proportional system all equilibria are such that each party gets a share of the vote greater that $\frac{1}{M}$. Then the introduction of a threshold $\frac{1}{M}$ *expands* the set of equilibria, weakly increasing the number of parties which may be observed in equilibrium.[7].

[7] As a remark, it should be observed that under a pure proportional system there may be equilibria in which a party gets a seat even if its share of the vote is less than $\frac{1}{M}$. For example, suppose that

With a majority premium, each party in the winning coalition will be overrepresented, i.e. it will get a share of seats greater than the share of the vote. We conjecture that this will lead to two effects. First, the different parties will tend to form minimal winning coalitions. Notice that there may be multiple equilibria; in some of them only two coalitions are formed, but in others there may be two main coalitions plus smaller coalitions or parties. Second, there should be a tendency to fragmentation inside each coalition and especially, in asymmetric situations, in the coalition which is more likely to win. This tendency is due to the fact that in the winning coalition each party gets more seats, and it should be stronger the stronger is the majority premium.

In general, the force that should keep a citizen from breaking a party is that this may favor coalitions different from the ones she belongs to.

Can we prove that, absent governmental rents, all coalitional equilibria are with minimally sized parties?

7.0.1 Governmental Rents

The conjecture is that government rents favor larger parties.

The reason is as follows. Suppose that I am just to the right or just to the left of x_2 in a three-party system. Then I am always part of the government and I get for sure a fraction of government rents

[MORE HERE].

If I form a different party then I am more likely to be excluded and thus I lose some government rents. Thus, even if in expectation my utility from policy is higher upon entry, I still prefer to stay out.

7.1 An Example: The Uniform Distribution

Under the uniform distribution there are multiple equilibria. We discuss here how both dominant party and maximum entry equilibria can be constructed.

Maximum entry. Suppose that M parties enter, with party i is located at $x_i = \frac{1}{2M} + \frac{i-1}{M}$. That is, the first party is positioned at $x_1 = \frac{1}{2M}$ and then each party

there are 10 seats, one party getting 55% of the vote and 5 parties getting 9% of the vote. Then under the Largest Remainder rule the largest party gets 5 seats and each of the smaller parties gets 1 seat. Of course this equilibrium would be destroyed by the introduction of the 10% threshold.

is located at a distance $\frac{1}{M}$ from the preceding party, up to the last party which is positioned at $x_M = 1 - \frac{1}{2M}$. Each party gets exactly $\frac{1}{M}$ of the vote and one seats. Notice that each party collects half of its votes on the right and half on the left. Clearly no party is better off exiting.

Consider entry. Any entrant at a position $y \in (x_i, x_{i+1})$ obtains a share of the vote

$$v_y = \frac{x_{i+1} + y}{2} - \frac{x_i + y}{2} = \frac{1}{2M}.$$

Since both x_i and x_{i+1} still have more than $\frac{1}{2M}$ of the vote, the entrant party obtains no representation. Furthermore, it does not change the distribution of seats[8].

Less than maximum entry. Proposition 4 states that in an equilibrium without a dominant party no party can get more than two seats. We now exhibit an equilibrium in which some party gets two seats. Suppose $M = 5$. There are 4 parties, located at x_1 =12.5, x_2 =37.5, x_3 =62.5 and x_4 =87.5. Thus, each party gets 25% of the vote and 1 seat, plus a second seat with probability $\frac{1}{4}$. Thus, one party ends up with two seats and the remaining three with one seat.

We want to show that entry is not profitable. Entry to the left of x_1 or to the right of x_4 yields less than 12.5% of the vote. Thus the entrant does not gain any seat and causes the closest party to have less than 25%; thus the lottery for the second seat will be among the parties which are more far away. Entry in the interval (x_1, x_2) gives 12.5% of the vote. Then x_1 and x_2 get exactly one seat. Parties 3 and 4 also receive one seat and the remaining seat is allocated with equal probability between the entrant and parties 3 and 4. For sufficiently low values of α this is not profitable.

Maximum entry with threshold $\frac{1}{M}$. When a threshold $\frac{1}{M}$ is introduced, a new maximum entry equilibrium appears in which extremists (parties located at 0 and 1) are allowed. Every running party gets exactly the threshold, therefore no other party can enter hoping to win seats. Also, entering to influence the policy is not good because a new party destroys the seats of the closest positions. In addition, just by introducing a an opportune threshold, equilibria with a smaller number of parties are possible. [**this part to be rewritten** *An example may help. Suppose there are four seats ($M = 4$) and uniform distribution. Without thresholds and dominant party, only an equilibrium with four parties running at $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}$ and $\frac{7}{8}$ is possible. If we introduce a threshold of 25% (that is $\frac{1}{m}$%) an equilibrium with three parties running*

[8]With Largest Remainder the parties located at x_i and x_{i+1} still retain their seat, since the other $M - 2$ parties have zero remainder. With d'Hondt, since both x_i and x_{i+1} get more that $\frac{1}{2M}$, no party collecting $\frac{1}{M}$ of the vote can get a second seat before one is assigned to both x_i and x_{i+1}.

at $\frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$ is possible as well.. Each incumbent gets 33% *of votes, but no new entrant will ever be able to obtain at least* 25% *of votes.*

This observation may suggest that when minimum threshold are applied at district level, having a threshold is more effective than reducing the size of the district in diminishing the number of parties running in that district.]

An equilibrium with a dominant party. Suppose there are 100 seats. Consider the following configuration: the dominant party is located at $x_1 = 0.405$, the opposition front runner is located at $x_2 = 0.615$ and all remaining parties located at $i + 0.005$, with $i = 0.62, ..., 0.99$. Then party 1 gets 51% and 51 seats, party 2 gets 11% and 11 seats and the remaining 37 parties get 1% of the vote and one seat each. We want to find for what values of α this is an equilibrium. In this equilibrium the policy 0.405 is implemented with probability 1. It is clear that no party is better off exiting, so we have to check that no entry is profitable. No entry to the right of x_2 is profitable, since the entrant would get less than 0.5% of the vote and no seats. Entry to the left of x_1, if successful in winning seats, causes a change in policy: instead of getting 0.405 with probability 1, the entrant gets a lottery. To simplify computations, let's approximate the share of seats with the share of votes. A party entering at $y < x_1$ gets a share $\frac{0.405+y}{2}$. Party one gets a share $\left(0.51 - \frac{0.405+y}{2}\right)$, while the shares of all other parties are unchanged. The expected utility from entry is

$$\alpha\left(\frac{0.405+y}{2}\right)-\left(0.51-\frac{0.405+y}{2}\right)(0.405-y)-0.11\,(0.615-y)-\frac{1}{100}\left(\sum_{i=62}^{99}\left(\frac{i}{100}+0.005-y\right)\right)$$

$$= 0.202\,5\alpha + \left(1 + \frac{1}{2}\alpha\right) y - \frac{1}{2}y^2 - 0.49999$$

while the expected utility from staying out is $-(0.405 - y)$. Thus, entry is unprofitable if

$$0.202\,5\alpha + \left(1 + \frac{1}{2}\alpha\right) y - \frac{1}{2}y^2 - 0.49999 < -(0.405 - y)$$

or

$$0.2025\alpha + \left(\frac{1}{2}\alpha\right) y - \frac{1}{2}y^2 < 0.094\,99$$

On the relevant range, the right hand side of this inequality is increasing in y. Thus, it is enough to check the inequality at $y = 0.405$. We get

$$0.2025\alpha + \left(\frac{1}{2}\alpha\right) 0.405 - \frac{1}{2}(0.405)^2 < 0.094\,99$$

or $\alpha < 0.43704$.

Finally, consider entry in the interval (x_1, x_2). If a new party enters in that interval the share of party 1 becomes $\frac{0.405+y}{2}$, while the share of party 2 becomes $\left(0.62 - \frac{0.615+y}{2}\right)$. The new party collects a share of the vote

$$v_y = \frac{0.615 + y}{2} - \frac{0.405 + y}{2} = 0.105.$$

Then the expected utility from entry becomes

$$\alpha 0.105 - \frac{y + 0.405}{2}(y - 0.405) - \left(0.62 - \frac{0.615 + y}{2}\right)(0.615 - y) - \frac{1}{100}\left(\sum_{i=62}^{99}\left(\frac{i}{100} + 0.005 - y\right)\right) =$$

$$= y + 0.105\,\alpha - y^2 - 0.417\,98$$

Since the utility which is attained in case of no entry is $-(y - 0.405)$, the condition for no entry to be the best strategy is

$$y + 0.105\,\alpha - y^2 - 0.417\,98 < y - 0.405.$$

If the inequality is satisfied at $y = 0.405$ then it is satisfied for each $y \in (0.405, 0.615)$. Thus the relevant inequality is

$$0.105\,\alpha < (0.405)^2 + 0.417\,98 - 0.405,$$

which is satisfied for $\alpha < 1.6858$.

8 Extensions

[TO BE WRITTEN]

In this section we discuss various extensions of the model.

- How are results modified for a positive (but not very large) c.
- A possible theoretical effect of thresholds that we should explore is the following. First, one possible equilibrium in which under proportional rule is one in which two parties govern and they win with sufficient margin that they can exclude smaller parties. This can be supported by a situation in which whenever a group of politicians form a new party they trigger entry by additional parties. Ex post, the large party can pick and choose among the small parties, excluding some

of them. If the likelihood of participating to the government is an important part of the politicians' motivations, the increased probability of being out of government may be a sufficient threat to avoid a breakaway in the first place, so the two-party equilibrium is sustained. Quite paradoxically, in such a situation it may well happen that imposing a threshold for representation may end up increasing the number of parties. Under such a threshold a breakaway party is shielded from the competition of smaller parties, since further fragmentation would bring all small parties below the threshold. This destroys the two-party equilibrium, since a group of politicians can form a party of a size around the threshold and become pivotal in the subsequent legislative game.

- The size of college districts.

9 Conclusions

[TO BE WRITTEN]
Some conclusions.

Appendix

Proof of Proposition 1. If no party runs then a citizen located at x can form a party, obtain a seat share $s_x = 1$ and implement the preferred policy. This gives a utility of $\alpha(1-c) > 0 \geq \underline{u}_x$, where $\underline{u}_x$ is the status quo utility for a citizen located at x.

Suppose $c < \widehat{s}_M$. In any equilibrium with a single party it must be optimal for all other citizens to stay out. However, no matter where the entering party is located, there is some entrant who can get a vote share arbitrarily close to 50% and thus a seat share of at least $\widehat{s}_M$. With $c < \widehat{s}_M$ this makes it optimal to enter for any positive value of α. This implies that there is no single party equilibrium.

Suppose now that $c \geq \widehat{s}_M$ and a single party is located at the median. Entry by a new party does not change the policy implemented, since the median party will retain more than 50% of the vote. The additional utility for the entrant is therefore $\alpha(s^e - c)$, where s^e is the share of seats of the entrant. Since $s^e \leq \widehat{s}_M$, entry is not profitable. ■

Proof of Proposition 2. Let $\{x_1, \ldots, x_J\}$ be the locations of the parties entering in equilibrium, m_i the number of seats and v_i the vote share obtained by the party located at x_i (which, for simplicity, will be referred to as party i). Assume that the party located at x_k is dominant so that it receives at least $\lfloor \frac{M}{2} \rfloor + 1$ seats.

We want to show that $m_k < \lfloor \frac{M}{2} \rfloor + 3$. Suppose not It must be the case that

$$\max\left\{F\left(\frac{x_k + x_{k+1}}{2}\right) - F(x_k), F(x_k) - F\left(\frac{x_k + x_{k+1}}{2}\right)\right\} \geq \frac{3}{M},$$

i.e. either the vote share collected on the right of x_k or the vote share collected on the left of x_k is enough to give 3 seats. Assume $F\left(\frac{x_k+x_{k+1}}{2}\right) - F(x_k) \geq \frac{3}{M}$ (the other case is easily adapted) and let $x^* \in \left(x_k, \frac{x_k+x_{k+1}}{2}\right)$ be the position such that, by winning a share $F(x^*) - F\left(\frac{x_k+x_{k-1}}{2}\right)$ of the vote party k obtains exactly $\lfloor \frac{M}{2} \rfloor + 1$ seats (with the convention $\frac{x_k+x_{k-1}}{2} = 0$ if $k = 1$).

Consider now a party entering at $y^* = 2x^* - x_k$. The vote share of this party is $F\left(\frac{x_{k+1}+y^*}{2}\right) - F(x^*)$. Now observe that $F\left(\frac{x_{k+1}+y^*}{2}\right) > F\left(\frac{x_{k+1}+x_k}{2}\right)$, and that the share of votes $F\left(\frac{x_k+x_{k+1}}{2}\right) - F(x^*)$ was enough for the dominant party to gain at least 2 additional seats. In order to gain two seats the additional share of the vote cannot be lower than $\frac{1}{M}$. This implies that a party entering at y^* can earn a seat without changing the policy and thus be better off, a contradiction. Thus, the dominant party has at most $\lfloor \frac{M}{2} \rfloor + 2$ seats.

Consider now a party which is not contiguous to the dominant party, i.e. its position is neither x_{k-1} nor x_{k+1}. For simplicity consider the party located at x_{k+2} (again, the argument for x_{k-2} is easily adapted). Suppose that the party obtains at least three seats. To achieve that it must get a share of at least $\frac{2}{M}$ of the vote. Thus, either $F\left(\frac{x_{k+2}+x_{k+3}}{2}\right) - F(x_{k+2})$ or $F(x_{k+2}) - F\left(\frac{x_{k+2}+x_{k+3}}{2}\right)$ are strictly greater than $\frac{1}{M}$. But this implies that a new party can enter either just to the right or just to the left of x_{k+2} and gain at least one seat without changing the policy. Thus any party which is not contiguous to the dominant party can't have more than two seats

The first point follows from the fact that otherwise a new party entering just to the right of x_{k+1} could get at least $\frac{1}{M}$ of the vote, thus one seat, without changing the policy implemented.

To prove the second point, suppose that the party has strictly more than two seats. Thus its share of the vote must be at least $\frac{2}{M}$. Furthermore, given point 1, it must be $F(x_{k+1}) - F\left(\frac{x_k+x_{k+1}}{2}\right) > \frac{1}{M}$. But, if the dominant party has more votes than what barely needed to get a majority, then a new party can enter just to the left of x_{k+1}, win a seat and leave the policy unchanged. This is a contradiction, so we conclude that whenever the DP gets more seats than what strictly needed to get a majority then the contiguous party must obtain a share of the vote lower than $\frac{2}{M}$, and thus at most two seats. Thus, an equilibrium in which a contiguous party obtains more than two seats is possible only when the dominant party gets a vote share which is barely enough to get an $\lfloor \frac{M}{2} \rfloor + 1$ seats. ■

Proof of Proposition 3. **TO BE completed.** The structure of any dominant party equilibrium must be such that no additional party wants to enter. This is always satisfied for citizen located to the right of x_{k+1} or to the left of x_{k-1}. Thus, a dominant party equilibrium can exist if it is possible to find an interval $[\underline{x}, \overline{x}]$ such that a party located inside the interval can become dominant without any other citizen inside the party being willing to enter.

Here we focus on the existence of a dominant left-wing party, and later generalize. For this case $\underline{x} = 0$.

Let $N^* = \lfloor \frac{M}{2} \rfloor$. The first requirement must be that the party gets at least a share $\frac{N^*}{M}$ of the vote, but less than $\frac{N^*+2}{M}$. Thus we have

$$F^{-1}\left(\frac{N^*}{M}\right) \leq \frac{x_1 + x_2}{2} \leq F^{-1}\left(\frac{N^* + 2}{M}\right)$$

Location x_1 must the to the left of $F^{-1}\left(\frac{N^*+1}{M}\right)$, since otherwise a new party could enter just to the left of x_1 and win N^*+1 seats.

Let now y be the entry in the interval $[0, x_1]$ which maximizes the share of the vote. Thus must in fact be the entry just to the left of x_1.

If all other parties get $\frac{1}{M}$ then entry at x_1 forces loss of the absolute majority. Let $s_{entr} = \frac{N_{entr}}{M}$ be the share of seats. Then the condition is

$$\alpha F(x_1) \leq \frac{1}{M} \sum_{j=2}^{J} (x_j - x_1)$$

$$\alpha \frac{N_{entr}}{J-1} + x_1 \leq \frac{\sum_{j=2}^{J} x_j}{J-1}$$

The maximum number of parties in a DP equilibrium is $M - N^*$. Assume for simplicity M is even, so $N^* = \frac{M}{2}$. Then the condition is

$$2\alpha \frac{N_{entr}}{M-2} + x_1 \leq \frac{2\sum_{j=2}^{\frac{M}{2}} x_j}{M-2}$$

Let m be the median. Then a sufficient condition is

$$2\alpha \frac{N_{entr}}{M-2} + x_1 \leq \frac{2m\left(\frac{M}{2}-1\right)}{M-2} = m$$

Thus, an equilibrium with a dominant party always exists provided that α is sufficiently small.

To see that for $\alpha > \frac{M}{2}$ no DP equilibrium can exist, observe that the dominant party gets at least half its seats either to the left or the right. In the current equilibrium the potential entrant gets zero. A party entering either to the left or the right gets at least $\frac{M}{4}$ seats. The condition is

$$\alpha \frac{1}{4} - \frac{1}{M} \sum_{j=2}^{J} (x_2 - x_1) \leq 0$$

$$\alpha \frac{M}{4} + (J-1)\, x_1 \leq \sum_{j=2}^{J} x_j$$

setting each $x_j = 1$ and $x_1 = 0$ we get

$$\alpha \frac{M}{4} \leq J - 1$$

$$\alpha \frac{M}{4} \leq \frac{M}{2}$$
$$\alpha \leq 2.$$

Thus, this is a necessary condition.

Notice that the maximum number of parties in a DP equilibrium is

$$M - N^*$$

When M is even then this is $\frac{M}{2}$. When M is odd then $N^* = \frac{M}{2} - 0.5$, so that the condition becomes

$$\alpha \leq 2\left(1 + \frac{1}{M}\right).$$

■

Proof of Proposition 4. Let $\{x_1, \ldots, x_J\}$ be an equilibrium without a dominant party. Let N_i be the number of seats won by party i and v_i its share of the vote. Furthermore, call v_i^- the share of the votes obtained by party i from citizen located on its left (i.e. citizens with ideal point $x < x_i$) and v_i^+ the share of votes obtained on the right.

Suppose that the party located at x_k gets the highest number of seats. We want to show that if $N_k \geq 3$ there is some citizen who can profitably form a party and enter.

If $N_k \geq 2$ then it must be the case that $v_k > \frac{1}{M}$ and $\max\left\{v_k^-, v_k^+\right\} > \frac{1}{2M}$. Without loss of generality, suppose $v_k^+ > \frac{1}{M}$. Then a party can enter just to right of x_k and get 1 seat, thus increasing utility by α. Let $v_{\min}$ be the share of the vote of smallest party obtaining representation.

Lemma

There must be at least one party with one seat.

Suppose not. Let us rank the parties in ascending vote order, with

$$v_{(1)} \leq v_{(2)} \leq \ldots \leq v_{(J)},$$

and assume that $N_{(1)} \geq 2$.

be the share of the vote of smallest party obtaining representation.

In addition to the seat going to the new entrant, at most one seat changes hands. Thus, the worst that can happen is that the probability distribution on policy increases by $\frac{1}{M}$ for the worst possible alternative and decreases by $\frac{1}{M}$ for the best possible alternative. Thus the maximum loss in utility is $\frac{1}{M}$. It follows that entry is profitable, a contradiction.

Suppose that a party is located exactly at 0. Then a citizen located at ε, for ε small enough, can enter and get a fraction $F\left(\frac{x_2+\varepsilon}{2}\right) - F\left(\frac{\varepsilon}{2}\right)$ of votes. Thus the distribution of votes after entry will be such that parties from 3 on still get $\frac{1}{M}$ of the vote, party 2 and the entrant get slightly less but very close to $\frac{1}{M}$ and party 1 gets zero. With both D'Hondt and Hare this means that the entrants gains a seat. Furthermore, the policy lottery improves for the entrant since, with probability $\frac{1}{M}$ the preferred policy (rather than zero) is implemented. **TO BE CHECKED AGAIN.**

■

Proof of Proposition 5. The first point is trivial. If a configuration $\{x_1, \ldots, x_k\}$ is an equilibrium under a pure proportional system then no other party can profitably enter. This remains true when the threshold is imposed. Similarly, if each existing party has a share of the vote greater than v, then the introduction of the threshold does not affect the outcome and thus it cannot make it profitable to exit.

Consider now the second point. Let $\frac{1}{v}$ be integer. Consider an equilibrium in which $n^* = \frac{1}{v}$ parties enter and their locations are obtained as a solution to the system of equations

$$F\left(\frac{x_i + x_{i+1}}{2}\right) - F\left(\frac{x_i + x_{i-1}}{2}\right) = v, \tag{2}$$

with $i = 1, \ldots, n^*$ and the conventions $F\left(\frac{x_1+x_0}{2}\right) = 0$, $F\left(\frac{x_{n^*}+x_{n^*+1}}{2}\right) = 1$. Each party obtains exactly $v\%$ of the vote and it obtains representation. No party is better off exiting, since this would cause a loss of rents and would change unfavorably the lottery on the implemented policy. No entrant can obtain representation: Any entering party would obtain strictly less than $v\%$ of the votes, and therefore no representation, and it would cause the closest parties to lose representation, thus affecting negatively the policy. Thus, the proposed configuration is an equilibrium.

■

References

[1] David Austen–Smith (2000) 'Redistributing Income under Proportional Representation', *Journal of Political Economy*, **108**: 1235-1269.

[2] David Austen-Smith and Jeffrey Banks (1988), 'Elections, Coalitions, and Legislative Outcomes', *American Political Science Review*, **82**: 405–422.

[3] Siddhartha Bandyopadhyay and Mandar Oak (2004) 'Party Formation and Coalitional Bargaining in a Model of Proportional Representation', FEEM working paper 98.2004.

[4] David Baron and Daniel Diermeier (2001) 'Elections, Governments, and Parliaments in Proportional Representation Systems', *Quarterly Journal of Economics*, **116**: 933–967.

[5] Jeffrey Banks and John Duggan (2006) 'A General Bargaining Model of Legislative of Policy-making', *Quarterly Journal of Political Science*, **1**: 49–85.

[6] André Blais and Marc André Bodet (2006) 'Does Proportional Representation Foster Closer Congruence Between Citizens and Policy Makers?', *Comparative Political Studies*, **39**: 1243–1262.

[7] Georges Casamatta and Philippe Donder (2005) 'On the influence of extreme parties in electoral competition with policy-motivated candidates', *Social Choice and Welfare*, **25**: 1-29.

[8] Gary Cox (1987) 'Electoral Equilibrium under Alternative Voting Institutions', *American Journal of Political Science*, **31**: 82-108.

[9] Gary Cox (1990) 'Centrifugal and Centripetal Forces in Electoral Systems', *American Journal of Political Science*, **34**: 903-935.

[10] Gary Cox (1997) *Making Votes Count*, CUP, Cambridge.

[11] Daniel Diermeier and Antonio Merlo (2004) 'An empirical investigation of coalitional bargaining procedures', *Journal of Public Economics*, **88**: 783–797

[12] Stefano Gagliarducci, Tommaso Nannicini and Paolo Naticchioni (2008) 'Electoral Rules and Politicians' Behavior: A Micro Test', IZA Discussion Paper No. 3348.

[13] Alan Hamlin and Michael Hjortlund (2000) 'Proportional Representation with Citizen Candidates', *Public Choice*, **103**: 205–230.

[14] Matias Iaryczower and Andrea Mattozzi (2008) 'Many Enemies, Much Honor? On the Competitiveness of Elections in Proportional Representation Systems', in *The Political Economy of Democracy*, BBVA, Spain.

[15] Matias Iaryczower and Andrea Mattozzi (2009) 'On the Nature of Competition in Alternative Electoral Systems', mimeo, California Institute of Technology.

[16] Lijphart, Arend, (1994) *Electoral Systems and Party Systems: A Study of Twenty-Seven Democracies, 1945-1990*. Oxford: Oxford University Press.

[17] Massimo Morelli (2004) 'Party Formation and Policy Outcomes under Different Electoral Systems', *Review of Economic Studies*, **71**: 829–853.

[18] Roger Myerson and Robert Weber (1993) 'A Theory of Voting Equilibria', *American Political Science Review*, **87**: 102–114.

[19] Ignacio Ortuño-Ortin (1997)'A Spatial Model of Political Competition and Proportional Representation', *Social Choice and Welfare* **14**: 427–438.

[20] Martin Osborne and Al Slivinski (1996) 'A Model of Political Competition with Citizen Candidates', *Quarterly Journal of Economics*, **111**: 65-96.

[21] Marco Rimanelli, Eds (1999) *Comparative Democratization and Peaceful Change in Single–Party Dominant Countries*, St. Martin's Press, New York.

[22] Rein Taagepera (2008), *Predicting Party Size*, Oxford University Press, New York.

[23] Rein Taagepera and Mart Laatsit (2007), 'The mean distribution of vote and seat shares of secondary parties: FTPT vs list PR', mimeo.